

TWISTED CONNECTED SUMS AND SPECIAL RIEMANNIAN HOLONOMY

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ABSTRACT. We give a new, connected sum construction of Riemannian metrics with special holonomy G_2 on compact 7-manifolds. The construction is based on a gluing theorem for appropriate elliptic partial differential equations. As a prerequisite, we also obtain asymptotically cylindrical Riemannian manifolds with holonomy $SU(3)$ building up on the work of Tian and Yau. Examples of new topological types of compact 7-manifolds with holonomy G_2 are constructed using Fano 3-folds.

The purpose of this paper is to give a new construction of compact 7-dimensional Riemannian manifolds with holonomy group G_2 . The holonomy group of a Riemannian manifold is the group of isometries of a tangent space generated by parallel transport using the Levi-Civita connection over closed paths based at a point. For an oriented n -dimensional manifold the holonomy group may be identified as a subgroup of $SO(n)$. If there is a structure on a manifold defined by a tensor field and parallel with respect to the Levi-Civita connection then the holonomy may be a proper subgroup of $SO(n)$ (it is just the subgroup leaving invariant the corresponding tensor on \mathbb{R}^n). There is essentially just one possibility for such holonomy reduction in odd dimensions, as follows from the well-known Berger classification theorem. This ‘special holonomy’ group is the exceptional Lie group G_2 and it occurs in dimension $n = 7$, when a metric is ‘compatible’ with the non-degenerate cross-product on \mathbb{R}^7 , see §1 for the precise definitions. The only previously known examples of compact Riemannian 7-manifolds with holonomy G_2 are due to Joyce who used a generalized Kummer construction and resolution of singularities [23, 24] (for non-compact examples see e.g. [4, 8, 9, 10, 11]). The compact 7-manifolds with holonomy G_2 in this paper are obtained by a different, connected-sum-like construction for a pair of non-compact manifolds with asymptotically cylindrical ends.

It is by now understood that the existence problem for the metrics with holonomy G_2 can be expressed as a non-linear system of PDEs for a non-degenerate differential 3-form φ on a 7-manifold. More precisely, a solution 3-form defines a torsion-free G_2 -structure and thus a metric, from the inclusion $G_2 \subset SO(7)$. The holonomy of this metric is in general only *contained* in G_2 . In fact, we shall obtain the solutions, torsion-free G_2 -structures, by proving a gluing theorem for pairs of manifolds with holonomy $SU(3)$, a maximal subgroup of G_2 . To claim that the holonomy on the resulting compact 7-manifold is exactly G_2 it suffices to verify a topological condition: that the 7-manifold has finite fundamental group.

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To implement a connected sum strategy we require, in the first place, a suitable class of non-compact, asymptotically cylindrical Riemannian manifolds with holonomy $SU(3)$. The point is that while the task of constructing asymptotically cylindrical metrics with holonomy G_2 in general is not likely to be any easier than the search for holonomy G_2 on compact manifolds, the metrics with holonomy $SU(3)$ are understood much better. The special holonomy $SU(3)$ naturally occurs on manifolds of *complex* dimension 3, for the Ricci-flat Kähler metrics. In [36, 37] Tian and Yau obtained a number of existence theorems for complete Ricci-flat Kähler metrics on quasiprojective manifolds. With some additional work, we are able to find simple topological conditions on these manifolds to further ensure that we obtain Ricci-flat Kähler metrics with holonomy $SU(3)$ and with the desired cylindrical asymptotic model at infinity. The result, Theorem 2.4, produces a non-compact analogue of Calabi–Yau 3-folds and may be of independent interest—from the point of view of Riemannian geometry and analysis, these are examples of manifolds with ‘added boundary at infinity’, or *b*-manifolds in the sense of Melrose [30].

The manifolds with holonomy $SU(3)$ that we obtain have real dimension 6. The Riemannian product with a circle then yields 7-manifolds with the same holonomy and with an end asymptotic to a half-cylinder. The cross-section of this half-cylinder is the Riemannian product of two circles and a complex surface of type K3 with a hyper-Kähler metric. A closed compact Riemannian 7-manifold may be constructed from a pair of these asymptotically cylindrical 7-manifolds by truncating the ends, cutting off to the cylindrical metric and identifying the boundaries via an orientation-reversing isometry. Thus the connected sum operation that we perform is not a usual one as the cross-section of the neck is not a sphere. The second and more important distinction is the non-trivial, ‘twisting’ isometry map identifying the two boundaries. The basic idea is that the map interchanges the two S^1 factors to avoid an infinite fundamental group of the compact 7-manifold. The map also interchanges the complex structures (using Torelli theorem) on the two K3 surfaces in such a way that the compact 7-manifold has a well-defined G_2 -structure compatible with that on the summands. The precise construction can be found in §4.

The G_2 -structure thus constructed on the compact 7-manifold can be regarded as an ‘approximate solution’ to the PDEs for torsion-free G_2 -structures. The setting achieved by the above steps falls under a general pattern of gluing the solutions of elliptic PDEs for special Riemannian structures on (generalized) connected sums. The gluing argument is carried out in §5, using the analytical techniques developed in [27] and previously in a special case in [16]. It is worth to point out that, with a careful choice of the Banach spaces of differential forms, our gluing is unobstructed. Therefore, we can be sure to solve the equation and obtain a metric with holonomy G_2 on our compact 7-manifold if only the initial approximate solution is well-defined.

The examples we found for the connected sum construction are discussed in general in §6 and some specific calculations are performed in §8. They originate from the class of compact complex 3-folds with Ricci-positive Kähler metric, known as Fano manifolds. Fano manifolds were extensively studied over the past decades in relation to problems in algebraic geometry and Kähler–Einstein metrics. Exploiting the theory of Fano manifolds and K3 surfaces, and also an appropriate extension of the deformation theory of complex

manifolds, we are able to prove a general Theorem 6.44 providing the required initial setting for the ‘twisted connected sum’ from any pair of algebraic families of smooth Fano 3-folds. This quite rapidly leads to many new topological types of compact manifolds of holonomy G_2 . The geometrical consequences concerning a distinguished class of minimal submanifolds, called coassociative submanifolds, of these new 7-manifolds will be investigated in a sequel [28] to this paper.

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1. TORSION-FREE G_2 -STRUCTURES

This section gives a short summary of some standard results on the Riemannian geometry associated with the group G_2 . Further details can be found in [20], [24] and [34].

The group G_2 may be defined as the group of linear automorphisms of the cross-product algebra on \mathbb{R}^7 , the vectors in \mathbb{R}^7 being interpreted as pure imaginary octonions. Any automorphism in G_2 necessarily preserves the Euclidean metric and so G_2 is precisely the group leaving invariant the differential 3-form φ_0 on \mathbb{R}^7 ,

$$\langle a \times b, c \rangle = \varphi_0(a, b, c),$$

encoding the cross-product multiplication. The Euclidean metric is completely determined by φ_0 , in particular, there is an explicit relation

$$(1.1) \quad 6\langle a, b \rangle \, d\text{vol} = (a \lrcorner \varphi_0) \wedge (b \lrcorner \varphi_0) \wedge \varphi_0.$$

The group G_2 is a closed proper (14-dimensional) subgroup of $SO(7)$ and thus a compact Lie group.

The Hodge star on \mathbb{R}^7 is G_2 -invariant, thus G_2 is also obtained as the stabilizer of the 4-form $\sigma_0 = *\varphi_0$ on \mathbb{R}^7 in the $GL_+(7, \mathbb{R})$ -action. The $GL_+(7, \mathbb{R})$ -orbits of both φ_0 and σ_0 are open in the space of all 3- and 4-forms on \mathbb{R}^7 respectively. ($GL_+(7, \mathbb{R}) \subset GL(7, \mathbb{R})$ denotes the subgroup of orientation-preserving linear isomorphisms.)

If M is an oriented 7-manifold then a smoothly varying non-degenerate cross-product on the tangent spaces defines a G_2 -structure on M . Equivalently, a G_2 -structure on M may be given by a 3-form φ whose value φ_p at each point $p \in M$ may be written as φ_0 with respect to some positively oriented tangent frame giving an isomorphism $T_p M \cong \mathbb{R}^7$. The space of all G_2 -structures on M therefore may be identified with a subset of differential 3-forms, equivalent to φ_0 at every point of M , and we shall denote this subset by $\Omega_+^3(M) \subset \Omega^3(M)$ (it is not a linear subspace).

Any G_2 -structure, being a special case of $SO(7)$ -structure, induces a metric on M (the formula (1.1) recovers this metric up to a conformal factor). Given a G_2 -structure φ on M , $\varphi + \chi$ is again a G_2 -structure whenever $\|\chi\|_{g(\varphi)} < \varepsilon$ in the sup-norm in the metric induced by φ . In this sense, $\Omega_+^3(M)$ is an open subset of $\Omega^3(M)$. The constant ε is determined by a calculation in \mathbb{R}^7 and is independent of φ or M .

There is a completely equivalent, dual description of G_2 -structures on M as 4-forms in $\Omega_+^4(M) \subset \Omega^4(M)$ modelled point-wise on σ_0 . That is, a 4-form is in $\Omega_+^4(M)$ precisely when it may be realized as σ_0 at any point of M , by choosing a positively oriented tangent frame and thus an identification of tangent space with \mathbb{R}^7 . Similarly to $\Omega_+^3(M)$ above, the set $\Omega_+^4(M)$ is open in $\Omega^4(M)$ in the sup-norm.

The equivalence of the descriptions of G_2 -structures as 3-forms and 4-forms is given by the map

$$\Theta : \varphi \in \Omega_+^3 \rightarrow *_{g(\varphi)}\varphi \in \Omega_+^4,$$

which is non-linear because of the dependence of the Hodge star on φ through the induced metric $g(\varphi)$.

A metric $g(\varphi)$ coming from a G_2 -structure φ does not necessarily have holonomy contained in G_2 . For that to be the case, the G_2 -structure φ must be *torsion-free* [34], Lemma 11.5, a condition expressed by the differential equations

$$(1.2) \quad d\varphi = 0 \quad \text{and} \quad d\Theta(\varphi) = 0.$$

For a compact 7-manifold M with a torsion-free G_2 -structure φ , the holonomy $\text{Hol}(g(\varphi))$ is exactly G_2 if and only if $\pi_1(M)$ is finite [23], II Proposition 1.1.1. We use the terms *G_2 -manifold* and *G_2 -metric* in this paper in the ‘strong’ sense to mean a Riemannian manifold (M, g) such that $\text{Hol}(g) = G_2$ (and likewise for other holonomy groups).

2. ASYMPTOTICALLY CYLINDRICAL MANIFOLDS WITH HOLONOMY $SU(3)$

If a torsion-free G_2 -structure φ is given on a product of a 6-manifold and a circle S^1 and induces a product metric then there is a parallel non-zero vector field tangent to the S^1 direction. Respectively, the holonomy of the metric on the 6-manifold is contained in a maximal subgroup $SU(3) \subset G_2$ of the automorphisms of the cross-product algebra \mathbb{R}^7 fixing a non-zero vector. In particular, the metrics with holonomy in $SU(3)$ on a 6-manifold give rise to solutions of (1.2). In this section we construct a class of such metrics.

A metric g on a real $2n$ -dimensional manifold W has holonomy in $U(n)$ if and only if that metric is Kähler. In particular, W then admits an integrable complex structure I orthogonal with respect to g and a closed $(1, 1)$ -form, the Kähler form ω , parallel with respect to g . Ricci curvature of a Kähler metric is equivalent to the curvature of the induced Hermitian connection on the canonical bundle K_W of $(n, 0)$ -forms, so g is Ricci-flat precisely when K_W is a flat Hermitian line bundle. The sections of K_W parallel with respect to g are holomorphic, hence closed, forms. If there is a globally defined parallel $(n, 0)$ -form Ω on W , sometimes called a ‘holomorphic volume form’, then the holonomy of g is contained in $SU(n)$. This will be the case e.g. when W is simply-connected. Then

$\Omega \wedge \Omega^*$ is a g -parallel (n, n) -form, hence a constant multiple of the volume form ω^n , where Ω^* denotes the $(0, n)$ -form complex conjugate of Ω .

The $SU(3)$ -metrics obtained in this section are asymptotically cylindrical Ricci-flat Kähler metrics on complex 3-dimensional manifolds. These manifolds are non-compact and may be given as $W = \overline{W} \setminus D$, where \overline{W} is a compact Kähler manifold containing a compact complex surface D . We shall make an additional assumption that D can be given as the zero set of a holomorphic coordinate z on \overline{W} and a tubular neighbourhood U of D in \overline{W} is diffeomorphic, as a real manifold, to $D \times \{|z| < 1\}$. With a change of variable $z = e^{-t-i\theta}$, $t > 0$, $\theta \in S^1$, W can be considered as a manifold with cylindrical end $U \setminus D$ asymptotic to $D \times \mathbb{R}_{>0} \times S^1$. We want to put on W a Ricci-flat Kähler metric asymptotic to a product Ricci-flat Kähler metric on the half-cylinder $D \times \mathbb{R}_{>0} \times S^1$.

The holonomy representation for a product metric on $D \times \mathbb{R}_{>0} \times S^1$ reduces to the holonomy representation for a metric on D . We shall be interested in the case when D is a simply-connected compact complex surface with $c_1(D) = 0$, that is, a K3 surface. It is a well-known consequence of Yau's solution of the Calabi conjecture [39] that a K3 surface admits a unique Ricci-flat Kähler metric in every Kähler class. We shall write κ_I for the Kähler form and $\kappa_J + i\kappa_K$ for a holomorphic volume $(2, 0)$ -form of this metric, where $\kappa_I, \kappa_J, \kappa_K$ are real 2-forms on D . The subscript notation indicates the fact that a Ricci-flat Kähler metric on a K3 surface is hyper-Kähler, compatible with the action of quaternions on tangent spaces, in particular one has $\kappa_I^2 = \kappa_J^2 = \kappa_K^2$. We shall consider the hyper-Kähler geometry of K3 surfaces in more detail in §4. For the moment, we only note that a product Ricci-flat Kähler metric on the half-cylinder $\mathbb{R}_{>0} \times S^1 \times D$ has Kähler form

$$(2.3a) \quad \omega_0 = dt \wedge d\theta + \kappa_I,$$

and holomorphic volume form

$$(2.3b) \quad \Omega_0 = (dt + id\theta) \wedge (\kappa_J + i\kappa_K).$$

The following theorem provides a class of manifolds carrying the $SU(3)$ -structures modelled on (ω_0, Ω_0) 'near infinity'. It builds up on the work of Tian and Yau [36, 37] on the existence of complete Kähler metrics with prescribed Ricci curvature. We shall sometimes refer to Kähler metrics by their Kähler forms. The notation $O(e^{-\lambda t})$ means that a function or differential form is bounded by $\text{const} \cdot e^{-\lambda t}$.

Theorem 2.4. *Let \overline{W} be a smooth compact Kähler 3-fold with $H^1(\overline{W}, \mathbb{R}) = 0$ and ω' the Kähler form on \overline{W} . Suppose that a K3 surface D in \overline{W} is an anticanonical divisor and has trivial self-intersection class $D \cdot D = 0$ in $H_2(\overline{W}, \mathbb{Z})$.*

Then $W = \overline{W} \setminus D$ admits an asymptotically cylindrical Ricci-flat Kähler metric g with respect to a diffeomorphism $U \setminus D \rightarrow D \times \{0 < |z| < 1\}$, where z is a holomorphic function on a neighbourhood U of D in \overline{W} and z vanishes to order 1 on D . The Kähler form of g can be written near D as

$$(2.5a) \quad \omega_g|_{U \setminus D} = \omega_0 + d\psi.$$

If g has a holomorphic volume form Ω_g then (up to a constant factor)

$$(2.5b) \quad \Omega_g|_{U \setminus D} = \Omega_0 + d\Psi.$$

Here ω_0, Ω_0 are identified with the differential forms on the (half-)cylinder given by (2.3) with $e^{-t-i\theta} = z$, and the Kähler class $[\kappa_I] \in H^{1,1}(D)$ is $[\omega'|_D]$. The 1-form ψ and 2-form Ψ are smooth and $O(e^{-\lambda t})$ with all derivatives (as measured by the metric ω_0), for any $\lambda < \min\{1, \sqrt{\lambda_1(D)}\}$, where $\lambda_1(D)$ is the first positive eigenvalue of the Laplacian on D with the metric κ_I .

Theorem 2.4 is proved in the next section.

Remark. There is a convenient way to formalize the idea of the ‘boundary at infinity’ of the Riemannian manifold (W, g) . A change of real coordinate $x = e^{-t}$, extended to map $t = \infty$ to $x = 0$, transforms W into a compact manifold \tilde{W} with boundary $S^1 \times D$ given as the zero set of x . The asymptotically cylindrical metric g given by Theorem 2.4 can be written near the boundary as

$$(2.6) \quad g = \left(\frac{dx}{x}\right)^2 + \tilde{g}.$$

Here \tilde{g} is a symmetric semi-positive bilinear form on \tilde{W} which is C^∞ -smooth up to the boundary and $\tilde{g}|_{x=0}$ is a product metric on $S^1 \times D$, with the Kähler metric κ_I on D . Elliptic theory on manifolds with boundary endowed with Riemannian metrics of type (2.6) is developed in [30], where these are called ‘exact b -metrics’. Results proved for manifolds with exact b -metrics are therefore valid, after a change of notation, on the asymptotically cylindrical manifolds obtained in Theorem 2.4. This will become important several times for the analysis in §3 and §5 below.

Examples of manifolds W satisfying the hypotheses of Theorem 2.4 will be given in §6. To ensure that they have a well-defined $SU(3)$ -structure (ω_g, Ω_g) we shall need the following.

Theorem 2.7. *Suppose that a 3-fold \overline{W} and K3 surface D in \overline{W} satisfy the hypotheses of Theorem 2.4 and let g be the asymptotically cylindrical Ricci-flat Kähler metric on $W = \overline{W} \setminus D$. If, in addition, \overline{W} is simply-connected and contains a complex curve ℓ with $D \cdot \ell = m > 0$ then the metric g has holonomy $SU(3)$.*

Proof. Consider \overline{W} as a union $\overline{W} = W \cup U$ where U is a small tubular neighbourhood of D . By the Seifert and van Kampen theorem (e.g. [1], Theorem 6.13), the fundamental group of \overline{W} may be expressed as the ‘amalgamated product’, $\pi_1(\overline{W}) = \pi_1(W) *_{\pi_1(W \cap U)} \pi_1(U)$, over $\pi_1(W \cap U)$. Since $\pi_1(U)$ is trivial, the amalgamated product in this case is just the quotient of $\pi_1(W)$ by the image of $\pi_1(W \cap U)$ under homomorphism induced by the inclusion $W \cap U \subset W$. By the hypothesis, this quotient is the trivial $\pi_1(\overline{W})$. Hence $\pi_1(W)$ is isomorphic to the quotient of $\pi_1(W \cap U) = \mathbb{Z}$ by a subgroup, so $\pi_1(W)$ is a *cyclic* group, generated by a loop around D in \overline{W} . Then $\pi_1(W) \cong H_1(W)$ with the coefficients in \mathbb{Z} . Furthermore, this cyclic group is *finite* as may be seen by considering a part of Mayer–Vietoris exact sequence for $\overline{W} = W \cup U$,

$$H_2(\overline{W}) \rightarrow H_1(U \setminus D) \rightarrow H_1(W) \oplus H_1(U) \rightarrow H_1(\overline{W}).$$

The first homomorphism maps to $H_1(U \setminus D) = \mathbb{Z}$ evaluating the intersection number with $[D]$ and its image contains $m\mathbb{Z}$ by the hypothesis. To simplify the notation we shall pretend that the image actually is $m\mathbb{Z}$, as the image is a non-trivial subgroup of \mathbb{Z} anyway. As $H_1(U) = H_1(\overline{W}) = 0$, we must have $\pi_1(W) \cong H_1(W) = \mathbb{Z}_m$.

If $m = 1$, so W is simply-connected, then there is nothing to prove as the vanishing Ricci curvature makes $\text{Hol}(g) = SU(3)$ the only possibility, by Berger classification of Riemannian holonomy groups, [34], Ch.10. For $m > 1$, we shall be done if we show that the flat Hermitian connection induced by g on the canonical bundle K_W is a trivial, product connection. It suffices to consider the holonomy transformation for this connection along a generator of $\pi_1(W)$. The loop around $D \subset \overline{W}$ corresponds to the S^1 factor in the cross-section of the cylindrical end of W . The holonomy transformation along this S^1 factor equals m th root of unity and is independent of the choice of loop in the homotopy class. On the other hand, as g decays to a product Ricci-flat metric on $S^1 \times D$, the same holonomy transformation tends to the identity when the loop goes to the infinity along the end of W . So, this transformation must be the identity and hence the holonomy of g is in $SU(3)$. Then by considering the universal Riemannian covering of W we conclude that $\text{Hol}(g)$ is precisely $SU(3)$. \square

3. PROOF OF THEOREM 2.4

Let $s \in H^0(\overline{W}, K_{\overline{W}}^{-1})$ denote the defining section for D . As the surface D has trivial self-intersection class in $H_2(\overline{W})$, the normal bundle of D has degree zero, by Mumford's self-intersection formula, [19], p.431. Then, by the adjunction formula, the anticanonical bundle $K_{\overline{W}}^{-1}$ restricts to a holomorphically trivial line bundle on D . With the help of a holomorphic trivialization of $K_{\overline{W}}^{-1}$ over a tubular neighbourhood U of D , the section s defines a holomorphic function, τ say, on U with $D = \tau^{-1}(0)$. The vanishing of Dolbeault cohomology $H^{0,1}(\overline{W})$ implies, cf. [18] pp.34–35, that $1/\tau$ can be extended to a meromorphic function on all of \overline{W} with the only pole along D of order 1. Thus τ extends to a K3 fibration of \overline{W} , which we denote again by $\tau : \overline{W} \rightarrow \mathbb{CP}^1$, with D a fibre. It also follows that $K_{\overline{W}}^{-1}$ is actually the pull-back via τ of a holomorphic line bundle of degree 1 over \mathbb{CP}^1 and $s = s_1 \circ \tau$ for a section s_1 over \mathbb{CP}^1 . We shall use z interchangeably to denote a holomorphic coordinate on \mathbb{CP}^1 and the corresponding pull-back holomorphic coordinate on \overline{W} . We can use the triviality of the normal bundle of D to define a local product decomposition

$$(3.8) \quad U \simeq \{|z| < 1\} \times D,$$

identifying $\tau^{-1}(z)$ with $\{z\} \times D$; note that (3.8) is only a diffeomorphism of the underlying *real* manifolds as the holomorphic structure of $\tau^{-1}(z)$ generally depends on z . (However z is holomorphic in both complex structures in (3.8).) The metric ω_0 defined in (2.3a) is Kähler with respect to the complex structure of $\{|z| < 1\} \times D$, $z = e^{-t-i\theta}$.

Recall that if ω is a Kähler metric on a complex n -manifold and $e^f \omega^n$ is the volume form of another Kähler metric then the Ricci form of the latter metric is $\text{Ric}(\omega) - i\partial\bar{\partial}f$. The

following existence result is a direct application of [36], Theorems 1.1 and 5.2 (cf. also [37], p. 52).

Proposition 3.9. *There is a choice of Kähler metric ω_{cyl} on W such that $\omega_{\text{cyl}}|_{U \setminus D}$ is commensurate with the cylindrical metric ω_0 and $\text{Ric}(\omega_{\text{cyl}}) = i\partial\bar{\partial}f$ with $\int_W (e^f - 1)\omega_{\text{cyl}}^3 = 0$, where $|f| = O(t^{-N})$ for some $N > 4$.*

Further, the complex Monge–Ampère equation

$$(3.10) \quad (\omega_{\text{cyl}} + \frac{i}{2\pi}\partial\bar{\partial}u)^3 = e^f \omega_{\text{cyl}}^3,$$

has a smooth solution u on W , such that on $U \setminus D \simeq \mathbb{R}_{>0} \times S^1 \times D$ the function u converges to zero uniformly in $S^1 \times D$, as $t \rightarrow \infty$. The derivatives of u are bounded (with respect to ω_{cyl}) and $\omega_{\text{cyl}} + \frac{i}{2\pi}\partial\bar{\partial}u > 0$. Thus $\omega_g = \omega_{\text{cyl}} + \frac{i}{2\pi}\partial\bar{\partial}u$ is a Ricci-flat Kähler metric on W commensurate with ω_{cyl} .

Proposition 3.11. *The solution u is uniquely determined by ω_{cyl} and f , where ω_{cyl} , f , and u are as defined in Proposition 3.9.*

Proof of Proposition 3.11. Suppose that u' is another solution of (3.10) and has the same properties as u , as given in Proposition 3.9. Recall that on a compact manifold the uniqueness of solution of (3.10), up to additive constant, is proved using local calculations and Stokes' Theorem to argue that the integral of $\Delta_g(u - u')^2$ is zero, where Δ_g denotes the Laplacian of the metric ω_g . See [39], Theorem 3, or [24], §5.7. The argument of the compact case remains valid in our setting because the solutions u, u' were assumed to decay to zero along the cylindrical end of W . \square

We claim that ω_g is the metric that we want, i.e. (2.5a) holds. We shall first deduce, by examining the method of [36], that the rates of decay of ω_{cyl} and f can be taken to be exponential in t . In the present situation we can exploit the K3 fibration structure of \overline{W} and the calculations are simplified.

Consider on \overline{W} a smooth function u_0 with support in U and such that $(\omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0)|_D$ is the Ricci-flat Kähler metric (Calabi–Yau metric) in the Kähler class $[\omega'|_D]$ on the K3 surface D . The form $\omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0$ is positive on the fibres of τ contained in some tubular neighbourhood of D . By smoothly cutting off to zero within this tubular neighbourhood, but away from D , the function u_0 may be chosen so that the form $\omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0$ is positive on any fibre of τ in \overline{W} .

Denote by ω_1 a Kähler form on $\mathbb{C}P^1$, such that $\omega_1 = (1 + O(|z|^2))idz \wedge d\bar{z}$ for small $|z|$, where $\tau^{-1}(\{z = 0\}) = D$. The pull-back $\tau^*\omega_1$ is a semi-positive (1,1)-form on \overline{W} , positive in the directions transverse to the fibres of τ . Rescaling ω_1 by an appropriate positive constant μ , we obtain a positive (1,1)-form

$$\omega_{\text{comp}} = \omega_{\text{comp}}(\mu) = \omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0 + \mu \tau^*\omega_1 \in \Omega^{1,1}(\overline{W}).$$

Recall that the section s defining D is the pull-back of a section s_1 over $\mathbb{C}P^1$. We can choose a Hermitian bundle metric over $\mathbb{C}P^1$ depending on a real parameter a , so that $e^{a/2}|s_1(z)| = |z|$, for small z , and the (1,1)-form $\Phi = \partial\bar{\partial}(\log|s_1|^2)^2$ on $\mathbb{C}P^1 \setminus \{z = 0\}$ is

positive on a punctured neighbourhood of $z = 0$, expressed there as $2(dz/z) \wedge (d\bar{z}/\bar{z})$. The ambiguity to choose the constant a will be needed later. The pull-back via τ defines a Hermitian bundle metric $|\cdot|_a$ on $K_{\overline{W}}^{-1}$, so that $|s(y)|_a = |s_1 \circ \tau(y)|$, $y \in \overline{W}$. Respectively, the $(1, 1)$ -form $\tau^*\Phi$ on $W = \overline{W} \setminus D$ is semi-positive near D and positive in the directions transverse to the fibres of τ near D .

With the above arrangements, define

$$(3.12) \quad \omega_{\text{cyl}} = \omega_{\text{comp}}(\mu) + \frac{i}{4}\tau^*\Phi = \omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0 + \mu\tau^*\omega_1 + \frac{i}{4}\partial\bar{\partial}(\log|s|_a^2)^2.$$

The last two terms in the right-hand side of (3.12) are pulled back from $\mathbb{C}P^1$ and it is not difficult to see that ω_{cyl} is a well-defined positive non-degenerate $(1, 1)$ -form, hence a Kähler form on W , for any sufficiently large constant $\mu > 0$. We note, for later use, that the section s can also be measured in the Hermitian bundle metric on $K_{\overline{W}}^{-1}$ induced by ω' . Near D this latter metric satisfies $|s(y)|^2 = e^{-\rho(y)}|s(y)|_a^2 = e^{-\rho(y)-a}|z|^2$, where ρ is a smooth function of y . Complementing z to a system of local holomorphic coordinates near a point in D , we obtain a local expression $\rho(x) + a = \log \det g'_{i\bar{j}}(x)$, where $g'_{i\bar{j}}$ are the local coefficients of ω' .

To deduce the asymptotic expression for ω_{cyl} , put $\kappa_I = (\omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0)|_D$ and extend κ_I to a closed real 2-form on U via the pull-back of the projection $U \rightarrow D$ along z , determined by the product decomposition $U \simeq \{|z| < 1\} \times D$. The inclusion $D \subset U$ is a homotopy equivalence, in particular, any 2-cycle in U is in the homology class of some 2-cycle in D . The class $[(\omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0)|_U - \kappa_I] \in H^2(U, \mathbb{R})$ then is trivial since it vanishes on every 2-cycle in D . Thus the restriction of the form $(\omega' + \frac{i}{2\pi}\partial\bar{\partial}u_0)$ to U may be written as $\kappa_I + d\nu$, for a smooth real 1-form ν on U with $(d\nu)|_D = 0$. We may use the product decomposition on U to expand in powers of z , writing

$$\nu = \nu_0 + f_0 dz + \bar{f}_0 d\bar{z} + O(|z|),$$

for some 1-form ν_0 and function f_0 on U pulled back from D , i.e. independent of z . The form ν is determined only up to addition of an exact 1-form (remember that $H^1(U, \mathbb{R}) = H^1(D, \mathbb{R}) = 0$). The property $(d\nu)|_D = 0$ implies $d\nu_0 = 0$, so ν_0 is exact and may be assumed zero. The last two terms in (3.12) are pulled back from $\mathbb{C}P^1$ and depend only on z . Direct calculation with the change of variable $e^{-t-i\theta} = z$ then shows that ‘near the divisor D at infinity’ the ω_{cyl} has the asymptotic form (2.5a) with $\lambda = 1$

$$(3.13) \quad \omega_{\text{cyl}}|_{U \setminus D} = dt \wedge d\theta + \kappa_I + d(e^{-t}\psi_{\text{cyl}}) = \omega_0 + d(e^{-t}\psi_{\text{cyl}}),$$

where ψ_{cyl} is a smooth 1-form bounded with all derivatives on the cylindrical end.

Remark. In fact, we can show by further calculation that ω_{cyl} can be chosen so that the last term in (3.13) has the decay rate e^{-2t} , rather than e^{-t} . However we omit these details as they are not needed for the later arguments.

The Ricci forms of the metrics ω' and ω_{cyl} are related by $\text{Ric}(\omega_{\text{cyl}}) - i\partial\bar{\partial}\log(\omega'^3/\omega_{\text{cyl}}^3) = \text{Ric}(\omega')$, and we can write

$$(3.14) \quad \text{Ric}(\omega_{\text{cyl}}) = i\partial\bar{\partial}f_{\text{cyl}}, \text{ where } f_{\text{cyl}} = -\log \frac{\omega_{\text{cyl}}^3}{\omega'^3} - \log|s|^2,$$

Here $|s|$ is taken in the metric induced by ω' and we used the standard expression $\text{Ric}(\omega') = -i\partial\bar{\partial}\log|s|^2$ for the Ricci form via the curvature form of K_W^{-1} . From (3.12) we have $\omega_{\text{cyl}}^3 = |z|^{-2}((i/2)dz \wedge d\bar{z} \wedge \omega_{\text{comp}}^2 + |z|^2\omega_{\text{comp}}^3)$ on the neighbourhood $U \setminus D$ in W when $|z| \neq 0$ is small. Using part of the standard argument making a Kähler metric ω_{comp} Euclidean to order 2 at a given point (e.g. [18], §0.7), we can complement the coordinate z near any point of D to local holomorphic coordinates z_0, z_1, z_2 , where $z_0 = z$, in such a way that $(i/2)dz \wedge d\bar{z} \wedge \omega_{\text{comp}}^2 = (1 + O(|z|^2)) \prod_{m=0}^2 (i/2)dz_m \wedge d\bar{z}_m$. On the other hand, in the same local coordinates $\omega'^3 = \det(g'_{i\bar{j}}) \prod_{m=0}^2 (i/2)dz_m \wedge d\bar{z}_m$. Using also the previous calculation $|s|^2 = |z|^2 \det(g'_{i\bar{j}})$ near D , we can write $|s|^2\omega_{\text{cyl}}^3/\omega'^3 = 1 + O(|z|^2)$. Passing to the cylindrical coordinates t, θ , we see, by inspection of (3.14), that the function f_{cyl} and any derivatives of f_{cyl} are bounded on W by a constant multiple of e^{-2t} . Here t is understood to be extended from U to a smooth non-negative function on W by cutting off to zero away from a neighbourhood of D .

In order to invoke the second part of Proposition 3.9 with $f = f_{\text{cyl}}$, it remains to show that we can satisfy the condition $\int_W (e^{f_{\text{cyl}}} - 1)\omega_{\text{cyl}}^3 = 0$. Note that as f_{cyl} decays exponentially to zero along the cylindrical end of W the integral is convergent. Recall that we have a choice of the real parameter a in the construction of $\omega_{\text{cyl}} = \omega_{\text{cyl}}(a)$ which does not affect the asymptotic properties achieved above. By (3.14) the summand $e^{f_{\text{cyl}}} \omega_{\text{cyl}}(a)^3 = \omega'^3|s|^{-2}$ is independent of a . On the other hand,

$$\begin{aligned} \partial\bar{\partial}(\log|s|_a^2)^2 &= \partial\bar{\partial}(a - \log|s|_0^2)^2 \\ (3.15) \quad &= \partial\bar{\partial}(\log|s|_0^2)^2 - 2a\partial\bar{\partial}\log|s|_0^2 = \partial\bar{\partial}(\log|s|_0^2)^2 - 2a\tau^*\Upsilon, \end{aligned}$$

where $\Upsilon = \partial\bar{\partial}\log|s_1|^2$ denotes the curvature form of the Hermitian line bundle over $\mathbb{C}P^1$ discussed earlier in this section. Calculating from (3.12) and (3.15),

$$\begin{aligned} \int_W (e^{f_{\text{cyl}}} - 1) \omega_{\text{cyl}}(a)^3 &= \int_W (\omega'^3|s|^{-2} - (\omega_{\text{cyl},0}^3 - 3\omega_{\text{cyl},0}^2 \frac{ai}{2} \tau^*\Upsilon)) \\ &= \int_W (\omega'^3|s|^{-2} - (\omega_{\text{cyl},0}^3 - \frac{3ai}{2} \omega'^2 \tau^*\Upsilon)), \end{aligned}$$

using Stokes' Theorem on \overline{W} for the latter equality, we find that $\int_W (e^f - 1) \omega_{\text{cyl}}(a)^3$ depends linearly on a . Note that $|s_1(z)| = e^{-a/2}|z|$ near $z = 0$ implies that the form $\tau^*\Upsilon$ is compactly supported in W . Further, the class $c_1(\overline{W}) = (2\pi i)^{-1}[\tau^*\Upsilon]$ of the anticanonical bundle of \overline{W} is semi-positive and ω' is positive non-degenerate and we deduce that $\int_W \omega'^2 \tau^*\Upsilon \neq 0$. Therefore, the integral $\int_W (e^{f_{\text{cyl}}} - 1) \omega_{\text{cyl}}(a)^3$ vanishes for some value of a .

Now, for the constructed (1,1)-form ω_{cyl} and the function f_{cyl} , there is a unique decaying solution u for the complex Monge–Ampère equation, according to Propositions 3.9 and 3.11.

Proposition 3.16. *The solution u satisfies $|\nabla^k u| < C_{k,\varepsilon} e^{-(\lambda-\varepsilon)t}$, for any $\varepsilon > 0$, where the constant $C_{k,\varepsilon}$ is independent of t . Here $\lambda = \min\{1, \sqrt{\lambda_1(D)}\}$ and $\lambda_1(D)$ is the smallest positive eigenvalue of the Laplacian on $C^\infty(D)$ for the metric κ_I .*

Before going to prove Proposition 3.16 we need to recall from [29],[30] some Fredholm theory for the Laplacian on asymptotically cylindrical Riemannian manifolds ‘with boundary at infinity’. (Note the remark on [30] after Theorem 2.4.)

We begin by introducing a framework of weighted Sobolev spaces for exponentially decaying functions. As before, let $t \in C^\infty(W)$ coincide along the cylindrical end of W with the \mathbb{R} -coordinate $t = -\log |z|$. Consider a smooth asymptotically cylindrical metric on W , more precisely, a metric which for the large values of t is asymptotic with all derivatives to a product metric on the cylindrical end $U \simeq \mathbb{R}_{>0} \times S^1 \times D$ of W (ω_{cyl} is an example of such metric). As t tends to infinity, the metric on $\{t\} \times S^1 \times D \subset W$ converges to some metric on $S^1 \times D$ and we shall call the Riemannian manifold $S^1 \times D$ with this limit metric the boundary of W at infinity. The weighted Sobolev space $e^{-\delta t} L_k^p(W)$ is the space of all functions $e^{-\delta t} v$ such that $v \in L_k^p(W)$. By definition, the norm of $e^{-\delta t} v$ in $e^{-\delta t} L_k^p$ is just the L_k^p -norm of v . The definition extends in the usual way to differential forms.

Proposition 3.17. (i) *Let W' be a smooth Riemannian manifold with a smooth asymptotically cylindrical metric and let q be an integer $0 \leq q \leq \dim W'$. Let $\delta > 0$ be such that δ^2 is not an eigenvalue of the Laplacian acting on $(\Omega^{q-1} \oplus \Omega^q)(\partial_\infty W')$ (on $C^\infty(\partial_\infty W')$ in the case $q = 0$), where $\partial_\infty W'$ denotes the boundary of W' at infinity. Then the Laplacian on the differential q -forms on W' defines, for every $p > 1$, $k \geq 2$, a bounded Fredholm operator $e^{-\delta t} L_k^p(\Omega^q(W')) \rightarrow e^{-\delta t} L_{k-2}^p(\Omega^q(W'))$. In particular, its image in $e^{-\delta t} L_{k-2}^p$ is closed.*

(ii) *Let W' be as in (i) and $\delta > 0$. Then the Laplacian Δ on the space of functions $e^{-\delta t} L_k^p(W')$ is injective. Given $\tilde{f} \in e^{-\delta t} L_{k-2}^p(W')$ with $\int_{W'} \tilde{f} = 0$, the equation $\Delta v = \tilde{f}$ has a (unique) solution $v \in e^{-\delta_1 t} L_k^p(W')$, where $\delta_1 \leq \delta$, $\delta_1 < \lambda_1$, $\lambda_1 > 0$, and λ_1^2 is the smallest positive eigenvalue of the Laplacian Δ_∂ on $\partial_\infty W'$. In particular, if $0 < \delta < \lambda_1$ then the image $\Delta(e^{-\delta t} L_k^p(W'))$ is a codimension-one subspace in $e^{-\delta t} L_{k-2}^p(W')$ of functions whose integral over W' is zero.*

Proof. The part (i) is a direct application of [29] or [30].

For the part (ii), we find by application of Theorem 7.4 of [29] that the index of the Laplacian on $e^{-\delta t} L_{k-2}^p(W')$ (for any $p > 1$, $k \geq 2$) is -1 whenever $0 < \delta < \lambda_1$. On the other hand, for any $\delta > 0$ the integration by parts is valid and proves that then any $e^{-\delta t} L_k^p$ function in the kernel of the Laplacian on W must be constant (cf. [30] p.224). As there are no non-zero constant functions in $e^{-\delta t} L_k^p(W')$ for $\delta > 0$, this gives the injectivity. The integration by parts also shows that the image $\Delta(e^{-\delta t} L_k^p(W'))$ consists of functions with zero integral over W' and the rest of (ii) then follows. \square

Remark. For the values $\delta > \lambda_1$ of the weight parameter, the index of the Laplacian will be strictly less than -1 , by direct application of the change of index formula in [29] or [30]. Then the Laplacian will no longer be onto the space $\{\tilde{f} \in e^{-\delta t} L_{k-2}^p(W') : \int_{W'} \tilde{f} = 0\}$ and so the decay estimate for v in Proposition 3.17(ii) cannot in general be improved.

Proof of Proposition 3.16. The complete Kähler metric ω_{cyl} on W , being an asymptotically cylindrical metric, has bounded curvature and injectivity radius bounded away from zero. Therefore, we still have Sobolev embedding $L_k^p(W) \subset C^r(W)$ for $r < k - 6/p$ (see [3] §2.7).

We shall be done if for some fixed $p > 1$ we show that $u \in e^{-(\lambda-\varepsilon)t} L_k^p(W)$, for any $\varepsilon > 0$ and any integer k .

Choose $p > 1$, such that $2 - 6/p > 1$, and let $\lambda - \varepsilon > 0$ be as in the hypothesis. The boundary of W at infinity is the Riemannian product $D \times S^1$, where the metric on D is κ_I and S^1 is the unit circle. The complete set of eigenfunctions of the Laplacian for the product metric is generated as products of the eigenfunctions on the factors ([6], p.144), so the eigenvalues of the Laplacian on $C^\infty(D \times S^1)$ are the sums $\lambda_j(D) + n^2$ of those on D and on S^1 . Thus $(\lambda - \varepsilon)^2$ is less than the first positive eigenvalue of the Laplacian of the product metric on $D \times S^1$. With the above choice of weight $\lambda - \varepsilon$ and Sobolev parameters p and $k \geq 2$, the integration by parts can be applied, so that there is a well-defined non-linear map

$$(3.18) \quad \begin{aligned} \mathcal{A} : e^{-(\lambda-\varepsilon)t} L_k^p(W) &\rightarrow \{\tilde{f} \in e^{-(\lambda-\varepsilon)t} L_{k-2}^p(W) : \int_W \tilde{f} \omega_{\text{cyl}}^3 = 0\}, \\ \mathcal{A}(v) &= \frac{(\omega_{\text{cyl}} + \frac{i}{2\pi} \partial \bar{\partial} v)^3}{\omega_{\text{cyl}}^3} - 1, \end{aligned}$$

which we further restrict to the open subset of those v satisfying $\omega_{\text{cyl}} + \frac{i}{2\pi} \partial \bar{\partial} v > 0$, so that $\omega_{\text{cyl}} + \frac{i}{2\pi} \partial \bar{\partial} v$ is a metric. The map \mathcal{A} is smooth with the derivative at v given by $(d\mathcal{A})_v = (3(\omega_{\text{cyl}} + \frac{i}{2\pi} \partial \bar{\partial} v)^3 / (2\pi \omega_{\text{cyl}}^3)) \Delta_v$ where Δ_v is the Laplacian associated to the metric $\omega_{\text{cyl}} + \frac{i}{2\pi} \partial \bar{\partial} v$ (cf. [3] §7.4). If v is in $e^{-(\lambda-\varepsilon)t} L_k^p(W)$ for any k , with some fixed p and ε , then $\omega_{\text{cyl}} + \frac{i}{2\pi} \partial \bar{\partial} v$ defines a smooth asymptotically cylindrical metric on W . In particular, the coefficient before Δ_v in the derivative of \mathcal{A} is smooth and asymptotically 1 along the cylinder. Also, by Proposition 3.17, the Laplacian Δ_v is injective and maps onto the target space of \mathcal{A} in (3.18) which is a closed subspace of the weighted Sobolev space $e^{-(\lambda-\varepsilon)t} L_{k-2}^p(W)$. Thus $(d\mathcal{A})_v$ is an isomorphism between Banach spaces $e^{-(\lambda-\varepsilon)t} L_k^p(W)$ and $\{\tilde{f} \in e^{-(\lambda-\varepsilon)t} L_{k-2}^p(W) : \int_W \tilde{f} \omega_{\text{cyl}}^3 = 0\}$.

We calculated that $\text{Ric}(\omega_{\text{cyl}}) = i\partial\bar{\partial}f$, where the function $f = f_{\text{cyl}}$ is in $e^{-(\lambda-\varepsilon)t} L_{k-2}^p(W)$ for any $p > 1$, $k \geq 2$, $\varepsilon > 0$, and $\int_W (e^f - 1) \omega_{\text{cyl}}^3 = 0$. If we could assume the norm of f in $e^{-(\lambda-\varepsilon)t} L_{k-2}^p(W)$ to be as small as we like then the Inverse Function Theorem in Banach spaces would give us a well-defined $\mathcal{A}^{-1}(e^f - 1) \in e^{-(\lambda-\varepsilon)t} L_k^p(W)$. Then the unique decaying solution of the complex Monge–Ampère equation given by Propositions 3.9 and 3.11 would be given by $u = \mathcal{A}^{-1}(e^f - 1)$, hence the required decay estimate.

Lacking the assumption that f is small, we start from the assertions of Proposition 3.9 that the solution u of $\mathcal{A}(u) = e^f - 1$ decays to zero along the cylindrical end of W and has bounded derivatives of any order. We next show that in fact all the derivatives of u decay to zero along the end of W . A suitable method may be adapted from the regularity arguments in [24] p.194, here is an outline. Consider the linear operator \mathcal{D} defined by

$$\mathcal{D}v = i\partial\bar{\partial}v \frac{\omega_u^2 + \omega_u \omega_{\text{cyl}} + \omega_{\text{cyl}}^2}{\omega_{\text{cyl}}^3},$$

where $\omega_u = \omega_{\text{cyl}} + \frac{i}{2\pi} \partial \bar{\partial} u$. Let $B(y, 2R) \subset W$ denote the geodesic ball about $y \in W$ of fixed, sufficiently small radius $2R$, in the metric ω_{cyl} . A positive $2R$ smaller than the injectivity radius of ω_{cyl} can be chosen independent of $y \in W$. Then \mathcal{D} induces, by restriction and with the help of the exponential mapping at y , an elliptic operator, \mathcal{D}_y say, on an open domain in \mathbb{C}^3 . Moreover, the lower bound on the eigenvalues of the matrix of coefficients of the principal symbol of \mathcal{D}_y and the upper bounds on the coefficients of \mathcal{D}_y on the domain in \mathbb{C}^3 may be taken independent of y . Applying a standard elliptic (Schauder) estimate on an open domain (e.g. [17], Theorems 6.2 and 6.17), putting $v = u$ and using $(2\pi)^{-1} \mathcal{D}u = \mathcal{A}(u) = e^f - 1$, we can show that

$$(3.19) \quad \begin{aligned} \|u\|_{C^{2+k}(B(y,R))} &< C(\|\mathcal{D}u\|_{C^k(B(y,2R))} + \|u\|_{C^0(B(y,2R))}) \\ &< C'(\|f\|_{C^k(B(y,2R))} + \|u\|_{C^0(B(y,2R))}). \end{aligned}$$

with the constants C, C' depending only on the metric ω_{cyl} and the norm of u in $C^{2+k}(W)$. The right-hand side of (3.19), as a function of $y \in W$, decays to zero along the end of W , so the derivatives of u decay as well. In particular, the metric ω_u is asymptotic, with all derivatives, to ω_{cyl} along the end of W .

Now the operator \mathcal{D} is asymptotic to $3\Delta_0$, the Laplacian of ω_{cyl} and, by the theory of [29] Sec.6, \mathcal{D} defines a Fredholm operator between the same $e^{-(\lambda-\varepsilon)t}$ -weighted Sobolev spaces as Δ_0 . Using the identity $(i\partial\bar{\partial}v)\omega_{\text{cyl}}^2 = (\Delta_0 v)\omega_{\text{cyl}}^3$ and similar ones for the metrics ω_u and $\omega_{\text{cyl}} + \omega_u$, we calculate $2(\mathcal{D}v)\omega_{\text{cyl}}^3 = (\Delta_0 v)\omega_{\text{cyl}}^3 + (\Delta_u v)\omega_u^3 + (\Delta_{0,u}v)(\omega_{\text{cyl}} + \omega_u)^3$. Then an integration by parts argument shows that \mathcal{D} is L^2 self-adjoint (in the metric ω_{cyl}), injective, has index -1 and the same cokernel as Δ_0 . As $\mathcal{D}u = 2\pi(e^f - 1) \in e^{-(\lambda-\varepsilon)t}L_{k-2}^p(W)$ we have $u \in e^{-(\lambda-\varepsilon)t}L_k^p(W)$, for all k . \square

This completes the proof of (2.5a).

Regarding (2.5b), note first that if Ω_g is a holomorphic volume form for ω_g then $z\Omega_g$ extends to a holomorphic $(3,0)$ -form on $U \subset \overline{W}$. On the other hand, $\Omega_0 = (-dz/z) \wedge (\kappa_J + i\kappa_K)$, so $z\Omega_0$ extends to a holomorphic $(3,0)$ -form in a different, product complex structure on the same neighbourhood $\{|z| < 1\} \times D \simeq U$. As the metric ω_g is asymptotic to ω_0 , we may without loss assume that $z\Omega_g = dz \wedge (\kappa_J + i\kappa_K)$ at any point in $\{0\} \times D$. The 3-forms Ω_g and Ω_0 are smooth on U and straightforward calculation in coordinates shows that $\Omega_g - \Omega_0$ and all derivatives are $O(e^{-t})$, hence certainly $O(e^{-(\lambda-\varepsilon)t})$ on the cylindrical end. We claim that $\Omega_g - \Omega_0 = d\Psi$ for some 2-form Ψ on $\mathbb{R}_{>0} \times S^1 \times D \simeq U \setminus D \subset W$. Indeed, for any 2-cycle $[C] \in H_2(D)$ the integral $\int_{\{t\} \times S^1 \times C} \Omega_g$ is independent of t by Stokes' Theorem, as Ω_g is closed. Letting t tend to infinity we find $\int_{\{t\} \times S^1 \times C} (\Omega_g - \Omega_0) = 0$, and since C was any 2-cycle we have $[\Omega_g - \Omega_0] = 0$ in $H^3(U \setminus D)$, so $\Omega_g - \Omega_0 = d\Psi$. For the decay properties of Ψ , extend Ψ smoothly from $U \setminus D$ to all of W , so that $d\Psi$ is in $e^{-(\lambda-\varepsilon)t}L_k^p\Omega^3(W)$ for any k and any $\varepsilon > 0$. We choose some $\lambda - \varepsilon > 0$ and may now apply to W Proposition 6.13 of [30] (cf. also the argument of Lemma 6.11 therein) to claim that the zero class $d\Psi$ of the 'usual' (C^∞) de Rham cohomology $H_{\text{dR}}^3(W)$ may be represented as the zero cohomology class of the $e^{-(\lambda-\varepsilon)t}$ -weighted Sobolev completion of the de Rham complex. Thus Ψ may be taken to be in $e^{-(\lambda-\varepsilon)t}L_{k+1}^p\Omega^2(W)$ (moreover, in $e^{-t}L_{k+1}^p\Omega^2(W)$), for every k , and the required decay property follows by Sobolev embedding.

4. THE COMPACT 7-MANIFOLDS

The Riemannian products of S^1 and the asymptotically cylindrical $SU(3)$ -manifolds of the previous section give real 7-dimensional manifolds with holonomy $SU(3)$. These asymptotically cylindrical 7-manifolds have torsion-free G_2 -structures, by the inclusion $SU(3) \subset G_2$, and can be ‘approximated’ (in a sense that we shall make precise below) by compact 7-manifolds with boundary. The boundary is the Riemannian product of $S^1 \times S^1$ and an $SU(2)$ -manifold underlying a Ricci-flat Kähler K3 surface. We want to take the union of two such 7-manifolds, with boundaries identified via an isometry, to produce a closed compact 7-manifold carrying a G_2 -structure with arbitrary small torsion.

A little thought shows that the naive identification of the boundaries is not a satisfactory option as the closed 7-manifold will then have an infinite fundamental group because of the S^1 factor, thereby violating a necessary condition for the existence of metrics of holonomy G_2 . We can avoid this topological obstruction by using a certain ‘twisted’ identification map. The construction of this map and of the family of ‘approximate solutions’ to the equations (1.2) is explained in this section.

We begin by defining a relation (and an isometry arising from that relation) for pairs of K3 surfaces. This uses some standard results on Kähler geometry of K3 surfaces which are only briefly recalled here, for a comprehensive reference see [5], Ch.VII.

Recall that the $SU(2)$ -metric giving the cylindrical asymptotic model in Theorem 2.4 is determined by a Ricci-flat Kähler K3 surface D . As we pointed out earlier, a Ricci-flat Kähler metric on a complex surface is hyper-Kähler; in terms of the holonomy groups this result corresponds to the low-dimensional coincidence $SU(2) = \mathrm{Sp}(1)$. Now a hyper-Kähler metric is characterized by the property that apart from the complex structure I inherited as a divisor in \overline{W} , D admits another two complex structures J and K satisfying the quaternionic relations $IJ = -JI = K$. The $SU(2)$ -metric is Kähler with respect to each of the I, J, K (and indeed with respect to any complex structure in the sphere $aI + bJ + cK$, $a^2 + b^2 + c^2 = 1$). The corresponding Kähler forms $\kappa_I, \kappa_J, \kappa_K$ are orthogonal to each other at every point of D and the volume form of the hyper-Kähler metric may be written as any of the $\kappa_I^2 = \kappa_J^2 = \kappa_K^2$. The form $\kappa_J + i\kappa_K$ is a holomorphic $(2, 0)$ -form with respect to I . Note that there is a complete $SO(3)$ -symmetry between I, J, K . In particular, given a Ricci-flat Kähler metric κ_I on D , the above normalizations determine $\kappa_J + i\kappa_K$ only up to a factor $e^{i\vartheta}$, $\vartheta \in \mathbb{R}$, i.e. there is an S^1 -ambiguity to choose the complex structure J and hence κ_J .

Denote by D_J the complex K3 surface defined by considering on D the complex structure J instead of I . Notice that (D, κ_I) and (D_J, κ_J) are isometric as the *real* 4-dimensional Riemannian manifolds but are in general *not* isomorphic as complex surfaces.

Definition. Let (D, κ_I) and (D', κ'_I) be two Ricci-flat Kähler K3 surfaces. We say that these satisfy the *matching condition* if there is a choice of holomorphic $(2, 0)$ -forms $\kappa_J + i\kappa_K$, $\kappa'_J + i\kappa'_K$, and an isomorphism of \mathbb{Z} -modules $h : H^2(D', \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z})$ preserving the cup-product and such that $h([\kappa'_I]) = [\kappa_J]$, $h([\kappa'_J]) = [\kappa_I]$, $h([\kappa'_K]) = -[\kappa_K]$, for the \mathbb{R} -linear extension of h .

Proposition 4.20. *Suppose that $(D, \kappa_I, \kappa_J + i\kappa_K)$ and $(D', \kappa'_I, \kappa'_J + i\kappa'_K)$ satisfy the matching condition. Then there is an isomorphism of complex surfaces $f : D_J \rightarrow D'$, such that $f^* = h$. Moreover, f is an isometry of hyper-Kähler manifolds, with the pull-back action on the Kähler forms given by*

$$(4.21) \quad f^* : \kappa'_I \mapsto \kappa_J, \quad \kappa'_J \mapsto \kappa_I, \quad \kappa'_K \mapsto -\kappa_K.$$

Proof. The first claim is a simple application of the (global) Torelli theorem for Kähler K3 surfaces. We have $H^{2,0}(D') = \mathbb{C}[\kappa'_J + i\kappa'_K]$, $H^{0,2}(D') = \mathbb{C}[\kappa'_J - i\kappa'_K]$ in the complex structure of D' , and $H^{2,0}(D_J) = \mathbb{C}[\kappa_I - i\kappa_K]$, $H^{0,2}(D_J) = \mathbb{C}[\kappa_I + i\kappa_K]$ in the complex structure of D_J . Hence the \mathbb{C} -linear extension $h_{\mathbb{C}}$ of the isometry h given by the matching condition converts Hodge decomposition $H^2 = H^{2,0} \oplus H^{1,1} \oplus H^{0,2}$ for D' into the one for D_J . As h maps Kähler class $[\kappa'_J]$ to a Kähler class $[\kappa_J]$ it satisfies the ‘effective Hodge isometry’ conditions and so, by [5], Theorem VIII.11.1, h arises as a pull-back $h = f^*$ of a uniquely determined biholomorphic map $f : D_J \rightarrow D'$.

The second claim follows as a Ricci-flat Kähler metric on a K3 surface is uniquely determined by its Kähler class. So the image $f^*(\kappa'_I)$ of the Ricci-flat Kähler metric on κ'_I has to be the Ricci-flat κ_J on D_J . \square

Let W be a complex 3-fold with an $SU(3)$ -structure defined by a Kähler form ω and a holomorphic volume form Ω . The ‘product’ G_2 -structure on the real 7-manifold $W \times S^1$ is expressed by the 3-form φ in $\Omega^3_+(W \times S^1)$ or its Hodge dual in $\Omega^4_+(W \times S^1)$, respectively,

$$(4.22a) \quad \varphi = \omega \wedge d\theta' + \text{Im } \Omega,$$

$$(4.22b) \quad *\varphi = \frac{1}{2}\omega \wedge \omega - \text{Re } \Omega \wedge d\theta',$$

where θ' is a standard parallel 1-form on S^1 . The G_2 -structure φ satisfies (1.2) and defines on $W \times S^1$ a metric whose holonomy is isomorphic to the holonomy $SU(3)$ of W . Suppose further that Theorem 2.4 holds for W with a product decomposition $\iota : \mathbb{R}_{>0} \times S^1 \times D \rightarrow W$ for the cylindrical end of W .

Recall from §1, that if φ is a G_2 structure then $\varphi + \chi$ defines another G_2 -structure whenever $\|\chi\|_{C^0} < \varepsilon$, in the metric defined by φ , where $\varepsilon > 0$ is an ‘absolute’ constant. Let $\alpha : \mathbb{R} \rightarrow [0, 1]$ denote a cut-off function, $\alpha(t) \equiv 0$ for $t \leq 0$ and $\alpha(t) \equiv 1$ for $t \geq 1$. Using the expressions (2.5) for the asymptotically cylindrical $SU(3)$ -structure on W , put

$$\omega_T = \omega - d(\alpha(t - T + 1)\psi),$$

$$\Omega_T = \Omega - d(\alpha(t - T + 1)\Psi).$$

Then, as ψ and Ψ decay to zero,

$$(4.23) \quad \varphi_T = \omega_T \wedge d\theta' + \text{Im } \Omega_T,$$

gives a well-defined G_2 -structure $\varphi_T \in \Omega^3_+(S^1 \times W)$ for every $T \geq T_0 > 1$, for large enough T_0 .

By the construction, $d\varphi_T = 0$. But, as the cutting-off construction of G_2 -structure changes the Hodge star of the induced metric, the form $\Theta(\varphi_T)$ is in general not closed and

the G_2 -structure φ_T has torsion. It is convenient to measure the torsion of φ_T using the 4-form

$$(4.24) \quad \tilde{\Theta}(\varphi_T) \stackrel{\text{def.}}{=} \Theta(\varphi_T) - (\tfrac{1}{2}\omega_T \wedge \omega_T - \text{Re } \Omega_T \wedge d\theta'),$$

so $d\tilde{\Theta}(\varphi_T) = d\Theta(\varphi_T)$. The form $(\tfrac{1}{2}\omega_T \wedge \omega_T - \text{Re } \Omega_T \wedge d\theta')$, is *not* the Hodge dual of φ_T in the metric defined by φ_T (it would be the Hodge dual in the product metric on $W \times S^1$, if one could assume that ω_T is a well-defined Kähler form of a metric on W). On the other hand, by comparing the definition of φ_T with (4.22) and (2.3) we see that $\tilde{\Theta}(\varphi_T)$ is supported in the region $\iota(D \times S^1 \times S^1 \times [T, T+1])$ where the cutting-off was performed. Moreover, we have.

Lemma 4.25. *For any $\varepsilon > 0$,*

$$\|\tilde{\Theta}(\varphi_T)\|_{L_k^p(W \times S^1)} < C_{p,k,\varepsilon} e^{-(\lambda-\varepsilon)T},$$

with a constant $C_{p,k,\varepsilon}$ independent of T , where $p > 1$, $k = 0, 1, 2, \dots$, the exponent $\lambda > 0$ for a given W is as defined in Theorem 2.4, and the norms are taken in the metric on $W \times S^1$ induced by φ_T .

Proof. This is deduced in a straightforward way from Theorem 2.4. Note that all the differential forms, metrics, and their Hodge stars, appearing in the definition of $\tilde{\Theta}(\varphi_T)$ decay like $O(e^{-(\lambda-\varepsilon)T})$ to the cylindrical model determined by ω_0, Ω_0 of (2.3) and the cut-off function is fixed. \square

Let now W_1, W_2 be two asymptotically cylindrical manifolds, each with an $SU(3)$ -structure satisfying the assertions of Theorems 2.4 and 2.7. For $i = 1, 2$, denote

$$M_{i,T} = (W_i \setminus \iota(D_i \times S^1 \times]T+1, \infty[)) \times S^1,$$

so $M_{i,T}$ is a compact Riemannian 7-manifold with boundary $\partial M_{i,T}$ diffeomorphic to $D_i \times S^1 \times S^1$. Suppose that Proposition 4.20 holds for the K3 surfaces $D = D_1$ and $D' = D_2$. Put on each $M_{i,T}$ the G_2 -structure $\varphi_{i,T}$ as in (4.23). Define a closed smooth 7-manifold M as a union of $M_{1,T}$ and $M_{2,T}$, with collar neighbourhoods of the boundaries identified using the diffeomorphism

$$(4.26) \quad \begin{aligned} D_1 \times S^1 \times S^1 \times]T, T+1[&\rightarrow D_2 \times S^1 \times S^1 \times]T, T+1[, \\ (y, \theta_1, \theta_2, T+t) &\mapsto (f(y), \theta_2, \theta_1, T+1-t) \end{aligned}$$

where f is the hyper-Kähler isometry of K3 surfaces which we obtained in Proposition 4.20. The G_2 -structure near the boundary of $M_{i,T}$ is equivalent to the ‘product’ torsion-free G_2 -structure on the cylinder $D \times S^1 \times S^1 \times \mathbb{R}$ arising from the $SU(2)$ -structure on D . The corresponding 3-form $\varphi_{(D)}$ is expressed, using (2.3) and (4.22a), as

$$(4.27) \quad \varphi_{(D)} = \kappa_I \wedge d\theta_1 + \kappa_J \wedge d\theta_2 + \kappa_K \wedge dt + d\theta_1 \wedge d\theta_2 \wedge dt.$$

The property (4.21) of f^* ensures that $\varphi_{(D)}$ is preserved by the diffeomorphism (4.26) and thus the compact manifold M has a well-defined orientation and a family φ_T , $T > T_0$, of G_2 -structures induced from those on M_1 and M_2 . (Here we abused notation and started to write φ_T for a G_2 -form on M , but this should not cause confusion.)

The diffeomorphism class of M is independent of T . We shall write M_T as a shorthand for M endowed with the 3-form φ_T (and the induced metric). One can think of the Riemannian manifold M_T as a generalized connected sum of $W_1 \times S^1$ and $W_2 \times S^2$ with the neck of length approximately $2T$. The following theorem summarizes the properties of M and φ_T that we achieved in the above construction.

Theorem 4.28. (i) One has $b_1(M) = b_1(W_1) + b_1(W_2)$. Further, $\pi_1(M) = \pi_1(W_1) \times \pi_1(W_2)$; in particular, M is simply-connected if W_1 and W_2 are so.

(ii) There exists T_0 such that $\varphi_T \in \Omega_+^3(M)$ for $T > T_0$. Then for any $\varepsilon > 0$,

$$(4.29) \quad \|\tilde{\Theta}(\varphi_T)\|_{L_k^p(M_T)} < 2C_{p,k,\varepsilon} e^{-(\lambda-\varepsilon)T}$$

in the metric defined by φ_T , where $C_{p,k,\varepsilon}$ is the constant appearing in Lemma 4.25. Here $0 < \lambda \leq 1$ is the smallest of the two exponential decay parameters (as in Theorem 2.4), for W_1 and W_2 .

Remark. The topological claims on M follow by, respectively, Mayer–Vietoris exact sequence and van Kampen theorem, applied to $M = M_{1,T} \cup M_{2,T}$.

5. THE GLUING THEOREM

When the fundamental group of a closed 7-manifold M is finite, any torsion-free G_2 -structure, i.e. a solution φ to the system of equations

$$(5.30) \quad d\varphi = 0, \quad d\Theta(\varphi) = 0,$$

will make M into a Riemannian manifold with holonomy G_2 , [24] pp.244–245. We have constructed, in the previous section, a family $\varphi_T \in \Omega_+^3(M)$ of G_2 -structure 3-forms satisfying the first of the equations (5.30) and having arbitrary small $d\Theta(\varphi_T)$ as T tends to infinity. This sets the scene for application of the perturbative methods to find solutions to (5.30). It is convenient to look for the solutions in the form $\varphi_T + d\eta$, for an unknown 2-form η .

The equations (5.30) are invariant under the action of the group of diffeomorphisms of M whose Lie algebra is given by the vector fields on M . Respectively, the infinitesimal action on a closed form φ_T is just the Lie derivative $d(X \lrcorner \varphi_T)$ in the direction of a vector field X . So *a priori* a solution η would be transverse to the subspace of all the 2-forms $X \lrcorner \varphi_T$ generated by vector fields on M_T .

A way to eliminate the infinite-dimensional symmetry of (5.30) and reduce the torsion-free condition $d\Theta(\varphi + d\eta) = 0$ to an elliptic equation in η has been worked out in [24], §10.3. It uses some special geometry of G_2 -structures and the local expression $\Theta(\varphi + d\eta) = \Theta(\varphi) + (d\Theta)_\varphi d\eta + R(d\eta)$ with the second order remainder $R(d\eta)$ satisfying a quadratic estimate

$$(5.31) \quad \|dR(\chi_1) - dR(\chi_2)\|_{L^p} < (C_1 + C_2 \|d\Theta(\varphi_T)\|_{L_1^p}) (\|\chi_1\|_{L_1^p} + \|\chi_2\|_{L_1^p}) \|\chi_1 - \chi_2\|_{L_1^p},$$

for $p > 7$ and with the constants C_1, C_2 independent of φ_T or the manifold. The result may be stated as follows.

Proposition 5.32 (cf. [24], Theorem 10.3.7). *Let M' be a compact Riemannian 7-manifold whose metric comes from a G_2 -structure defined by a smooth closed form $\varphi' \in \Omega_+^3(M)$. Denote by $\langle \cdot, \cdot \rangle$ the induced inner product on the fibres of vector bundles associated to TM' .*

There exist constants $\rho > 0$, $\tilde{\rho} > 0$ independent of M' or φ' and such that if a 4-form $\tilde{\Theta}$ on M' , satisfies $d\tilde{\Theta} = d\Theta(\varphi')$ and $\|\tilde{\Theta}\|_{C^0} < \tilde{\rho}$, then for any 2-form η on M' , with $\|\eta\|_{C^1} \leq \rho$, the equation

$$(5.33) \quad (dd^* + d^*d)\eta + *d((1 + \frac{1}{3}\langle d\eta, \varphi' \rangle)\tilde{\Theta}) - *dR(d\eta) = 0,$$

implies $d\Theta(\varphi' + d\eta) = 0$.

In view of the estimate (4.29) the condition $\|\tilde{\Theta}(\varphi_T)\|_{C^0} < \tilde{\rho}$ in the above proposition will be satisfied for every sufficiently large T . Thus we can now turn to solving (5.33) for an unknown small η putting $\varphi' = \varphi_T$ and $\tilde{\Theta} = \tilde{\Theta}(\varphi_T)$ on the compact 7-manifold $M' = M_T$.

The terms involving second order derivatives of η in (5.33) are the Laplacian term, the linear term in η with coefficients controlled by $\tilde{\Theta}$, and the quadratic term which may be regarded as a linear operator whose coefficients depend on η and are bounded by $\sup_{M_T} |d\eta|$. This tells us that (5.33) is an elliptic problem, at least for a C^1 -small η and with T assumed large. If $d\eta$ is Hölder continuous and small in $C^0(M_T)$ then elliptic regularity arguments show that a weak, L_2^p solution η of (5.33) is in fact smooth. (We do not include the details for this latter claim but see [24] p.303, cf. also [39] p.363.) Recall that for $\varphi_T + d\eta_T$ to be a well-defined G_2 -structure, the term $d\eta_T$ must be small in the sup-norm on M_T . If we are to solve the second order equation (5.33) for η in Sobolev space L_2^p , then we need an embedding $L_2^p \subset C^1$ in dimension 7, which requires $p > 7$. In what follows we fix such a choice of p .

The existence of torsion-free G_2 -structures on M will be established by the following technical result.

Theorem 5.34. *There exist $T_0 > 0$ such that for each $T \geq T_0$ the equation (5.33) has a unique smooth solution 2-form η_T on M_T satisfying $\|\eta_T\|_{L_2^p} < K_{p,\delta}e^{-(\lambda-\delta)T}$ and $\|\eta_T\|_{C^1} < K_\delta e^{-(\lambda-\delta)T}$, for any positive δ , where $K_{p,\delta}, K_\delta$ are constants independent of T . Here λ is a constant determined in Theorem 4.28.*

Theorem 5.34 may be viewed as an instance of a gluing theorem for solutions of non-linear elliptic PDEs. We should remark right away that the equation (5.33) for torsion-free G_2 -structures was solved in [24], §§11.6-11.8 in a different geometrical setting. In principle, our gluing theorem could be deduced by verifying that the G_2 -structures φ_T with small torsion can be written so that they formally satisfy the hypotheses of Theorem 11.6.1 in [24] and then invoke that theorem to claim the existence of solutions to (5.33).

We shall prove Theorem 5.34 via a different, more direct and geometrically natural approach tailored to generalized connected sums. Another reason for our choice is that this proof follows up the applications of elliptic theory and Hodge theory on Riemannian manifolds with asymptotically cylindrical ends, already required in §3. We also are able to explicitly eliminate the issue of the ‘obstruction spaces’ (whose vanishing is sufficient for the existence of a solution) in this gluing problem.

The remainder of this section deals with the proof of Theorem 5.34. The following result is a standard variant of the contraction mapping principle and is included here for the reader's convenience and to keep track of the relations between various constants in the estimates.

Proposition 5.35. *Let $a : \mathcal{E} \rightarrow \mathcal{F}$ be a smooth map between Banach spaces, $a(\eta) = a_0 + A\eta + Q(\eta)$. Suppose that A is a linear isomorphism between \mathcal{E} and \mathcal{F} and that for all $\eta_1, \eta_2 \in \mathcal{E}$*

$$(5.36) \quad \|A^{-1}Q(\eta_1) - A^{-1}Q(\eta_2)\| \leq m(\|\eta_1\| + \|\eta_2\|)\|\eta_1 - \eta_2\|.$$

If

$$(5.37) \quad \|A^{-1}a_0\| < 1/(8m)$$

then the equation $a(\eta) = 0$ has a unique solution η_0 of norm less than $1/(4m)$. This solution also satisfies $\|\eta_0\| \leq 2\|A^{-1}\| \cdot \|a_0\|$.

In our situation, $a_0 = -*d\tilde{\Theta}(\varphi_T)$ is exponentially small in T , as $T \rightarrow \infty$, by Theorem 4.28(ii). Hence for large T the term $C_2\|d\Theta(\varphi_T)\| = C_2\|d\tilde{\Theta}(\varphi_T)\|$ in the quadratic estimate (5.31) of $Q(\eta) = -*dR(d\eta)$ can be absorbed by taking a larger constant C_1 independent of T . Furthermore, the linearization $A = A_T$ of (5.33) is the Laplacian on M_T perturbed by adding a linear differential operator $\frac{1}{3}*d(\langle d\eta, \varphi_T \rangle \tilde{\Theta}(\varphi_T))$ whose coefficients are exponentially small in T . Therefore, for every sufficiently large T , the operator A_T will have an elliptic symbol and will be invertible on the orthogonal complement $(\mathcal{H}^2)^\perp$ of harmonic 2-forms on M_T . As we are really interested in the value of $d\eta$, rather than η , and as all the terms in the left-hand side of (5.33) take values in $(\mathcal{H}^2)^\perp$, we may consider (5.33) as an equation in $(\mathcal{H}^2)^\perp$. The main issue in the application of Proposition 5.35 to (5.33) is to give a suitable upper bound on the operator norm of A_T^{-1} for any large T , so as to satisfy (5.36) and (5.37) together.

Our estimate for the linearized problem on M_T will be obtained by application of the technique developed in [27]. The basic principle is that the elliptic differential operator A_T may be obtained by gluing together two elliptic operators A_j on the non-compact manifolds $W_j \times S^1$, $j = 1, 2$ at their asymptotically cylindrical ends. More precisely, the coefficients of the operator A_T depend smoothly on the G_2 -structure φ_T on M_T , and the operators A_j are determined, in just the same way, by the G_2 -structures which we used in the construction of φ_T . As each of the non-compact manifolds $W_j \times S^1$ was given a torsion-free G_2 structure, we may take $\tilde{\Theta}_j = 0$, so the associated linear operators A_j are just the Laplacians for the respective metrics. We define

$$\begin{aligned} A_1 &= dd^* + d^*d : e^{-\delta t_1} L_2^p \Omega^2(W_1 \times S^1) \rightarrow e^{-\delta t_1} L^p \Omega^2(W_1 \times S^1), \\ A_2 &= dd^* + d^*d : e^{\delta t_2} L_2^p \Omega^2(W_2 \times S^1) \rightarrow e^{\delta t_2} L^p \Omega^2(W_2 \times S^1), \end{aligned}$$

where $\delta > 0$ and, as before, $t_j \in C^\infty(W_j \times S^1)$ denotes a real parameter along the cylindrical end. From Proposition 3.17(i), each A_j is a bounded Fredholm operator if the weight parameter $\delta > 0$ is small. We shall further need the following.

Proposition 5.38. *For every sufficiently small $\delta > 0$, (i) the Laplacian A_1 is injective, and (ii) every non-zero element of the kernel of the Laplacian A_2 is smooth and has an asymptotic expansion as $t_2 \rightarrow \infty$ with the leading term polynomial in t_2 .*

Proof. This is an application of a general result on the asymptotic expansions proved in [30] Proposition 5.61. The leading term of asymptotic expansion of an $e^{\varepsilon t} L_k^p$ kernel element may be written as $e^{-\mu t_j} \eta(t_j, y)$, where $-\mu < \varepsilon$ and a form $\eta(t, y)$ depends polynomially on t_j and smoothly on the coordinates y of the cross-section of the ends of $W_j \times S^1$. Further, μ^2 is an eigenvalue of the Laplacian (on differential forms) on the cross-section at infinity (pp.224–225 op.cit.). The cross-sections of $W_j \times S^1$ at infinity are isometric; choose $\delta > 0$ with δ^2 smaller than the first positive eigenvalue of the respective Laplacian.

For $\varepsilon = -\delta$, it follows that $\text{Ker } A_1$ coincides with the L^2 -kernel. It is proved in [2], Proposition 4.9, and [30], Proposition 6.14, that the L^2 -kernel of the Laplacian on differential forms on an asymptotically cylindrical manifold is isomorphic to the image of the natural homomorphism $H_c^* \rightarrow H^*$ of the de Rham cohomology of this manifold, where c indicates the cohomology with compact support. As $H_c^1(W_j, \mathbb{R})$ and $H^1(W_j, \mathbb{R})$ vanish, it suffices to consider the image of $H_c^2(W_j, \mathbb{R}) \rightarrow H^2(W_j, \mathbb{R})$. The latter map is a part of exact sequence $H_c^2(W_j, \mathbb{R}) \rightarrow H^2(W_j, \mathbb{R}) \rightarrow H^2(D \times S^1, \mathbb{R})$. The map of $H^2(W_j, \mathbb{R})$ is equivalent to the composition of injective homomorphisms $H^2(W_j, \mathbb{R}) \rightarrow H^2(\overline{W}_j, \mathbb{R}) \rightarrow H^2(D, \mathbb{R})$, the first one from the Mayer–Vietoris sequence, as in Theorem 2.7, the second by Lefschetz hyperplane theorem. (Note that $H^2(D \times S^1, \mathbb{R}) \cong H^2(D, \mathbb{R})$.) It follows that $H_c^2(W_j \times S^1, \mathbb{R}) \rightarrow H^2(W_j \times S^1, \mathbb{R})$ maps to zero and A_1 is injective.

If $\varepsilon = \delta$ then, by the same argument as above, a non-zero 2-form in the kernel of A_2 cannot be in $e^{-\delta t_2} L_2^p$. So, by the choice of δ , the only possibility for the leading exponent in the asymptotic expansion of such 2-form is $\mu = 0$. \square

We now define the weight function $w_T \in C^\infty(M_T)$. Let $-T < t < T$ denote the real parameter along the neck of M_T , increasing towards the $(W_2 \times S^1)$ -piece, and with $t = 0$ in the middle of the cutting-off region (recall §4). The w_T coincides along the neck with $e^{-\delta t}$ and is smoothly cut off to the constants $e^{\pm \delta T}$ away from the neck, so that $e^{-\delta T} \leq w_T \leq e^{\delta T}$ at every point in M_T . Thus w_T interpolates between the weights on $W_j \times S^1$. The weighted $w_T L_k^p$ -norms on a compact M_T are commensurate, for any T , to the usual L_k^p Sobolev norms, however the estimates depend on T , $e^{-\delta T} \|\eta\|_{p,k} \leq \|w_T \eta\|_{p,k} \leq e^{\delta T} \|\eta\|_{p,k}$. Put $\mathcal{E}_T = (\mathcal{H}^2)^\perp \cap L_2^p \Omega^2(M_T)$, $\mathcal{F}_T = (\mathcal{H}^2)^\perp \cap L^p \Omega^2(M_T)$, but use the *weighted* norms on these Banach spaces, $\|\eta\|_{\mathcal{E}_T} = \|w_T \eta\|_{L_2^p}$, $\|\eta\|_{\mathcal{F}_T} = \|w_T \eta\|_{L^p}$. Finally, define

$$(5.39) \quad A_T : \eta \in \mathcal{E}_T \rightarrow (dd^* + d^*d)\eta + \frac{1}{3} * d(\langle d\eta, \varphi_T \rangle \tilde{\Theta}(\varphi_T)) \in \mathcal{F}_T.$$

Proposition 5.40. *Suppose that the weight parameter $\delta > 0$ is small, so that the Laplacians A_j are Fredholm operators and satisfy the assertions of Proposition 5.38. Then there exists $T_* > 0$ such that for any $T > T_*$ the inverse of the operator A_T defined by (5.39) exists and satisfies $\|A_T^{-1}\| < Ge^{\delta T}$ with a constant G independent of T .*

Proof. We checked above that the A_T is invertible for every sufficiently large T . It suffices to prove that there exist constants G, T_* , such that $\|A_T \eta\| > Ge^{-\delta T} \|\eta\|$, for $\eta \in \mathcal{E}_T$, whenever

$T > T_*$. The proof goes by contradiction. Assume that Proposition 5.40 is *not* true, so there exists a sequence $\eta_n \in \mathcal{E}_{T_n}$, with $\|\eta_n\| = 1$, $T_n \rightarrow \infty$, such that $e^{\delta T_n} \|A_{T_n} \eta_n\| \rightarrow 0$.

This is an example of the situation considered in [27] §4.1; the only difference is that in the present application we do not avoid the ‘asymptotic kernels and cokernels’ and claim a slightly weaker estimate. Let $\beta_n = \beta \circ t_n \in C^\infty(M_T)$, where t_n is the real parameter along the neck of M_{T_n} as above, and β be a cut-off function equal to 0 on $(-\infty, 1]$ and to 1 on $[2, \infty)$. The arguments of Proposition 4.2 and Lemma 4.7 of [27] yield

$$(5.41) \quad \|(1 - \beta_n)\eta_n\| \rightarrow 0 \quad \text{and} \quad e^{\delta T_n} \|A_2(\beta_n \eta_n)\| \rightarrow 0,$$

Here for the first statement we also took into account that the operator A_1 has no kernel. In the second statement, we identified $\beta_n \eta_n$ with a form on $W_2 \times S^1$.

Let $\beta_n \eta_n = \eta'_n + \eta''_n$, respective to the L^2 -orthogonal decomposition $\text{Ker } A_2 \oplus (\text{Ker } A_2)^\perp$ of the domain of A_2 . Then $\eta''_n \rightarrow 0$ from (5.41). Also, since the kernel of A_2 is finite-dimensional, we may assume, passing to a subsequence if necessary, that $\eta'_n \rightarrow \eta_0 \in \text{Ker } A_2$. In view of the asymptotic expansion found in Proposition 5.38, we have $\|e^{-\delta t_2} \eta_0\|_{L^p_2\{t_2 > T\}} \geq C_0 e^{-\delta T}$, with C_0 independent of $T > T_*$ and $C_0 = 0$ if and only if $\eta_0 = 0$. Using finite dimension of $\text{Ker } A_2$ again, we can estimate by standard methods $\|e^{-\delta t_2} \eta'_n\|_{L^p_2\{t_2 > T\}} \geq C'_0 \|e^{-\delta t_2} \eta_0\|_{L^p_2\{t_2 > T\}}$, with $C'_0 \neq 0$ independent of $T > T_*$. It follows that

$$\|A_2(\beta_n \eta_n)\| = \|A_2(\eta''_n)\| \geq C \|e^{-\delta t_2} \eta''_n\|_{L^p_2\{t_2 > T_n\}} = C \|e^{-\delta t_2} \eta'_n\|_{L^p_2\{t_2 > T_n\}} > C C'_0 C_0 e^{-\delta T_n},$$

since $\beta_n \eta_n = 0$ on $\{t_2 > T_n\}$ and where $C \neq 0$. Then $C_0 = 0$ from (5.41), hence $\eta_0 = 0$ and $\beta_n \eta_n \rightarrow 0$. This leads to a contradiction, $1 = \|\eta_n\| \leq \|(1 - \beta_n)\eta_n\| + \|\beta_n \eta_n\| \rightarrow 0$, which proves Proposition 5.40. \square

We can replace the L^p_k norms in the quadratic estimate (5.31) by the weighted $w_T L^p_k$ norms, at the cost of weakening the right-hand side by the factor $e^{-3\delta T}$. Likewise, $\|d\tilde{\Theta}(\varphi_T)\|_{\mathcal{F}_T} \leq e^{\delta T} \|d\tilde{\Theta}(\varphi_T)\|_p$ and from Proposition 5.40 the action of A_T^{-1} amounts to an extra factor $e^{\delta T}$ in the estimates. We find that Proposition 5.35 applies with $\mathcal{E} = \mathcal{E}_T$, $\mathcal{F} = \mathcal{F}_T$, $A = A_T$, and $m = \text{const} \cdot e^{4T\delta}$ to solve (5.33) for every large T , provided that $0 < 6\delta < \lambda$, where λ is the constant in (4.29). This completes the proof of Theorem 5.34.

6. FROM FANO 3-FOLDS TO G_2 -MANIFOLDS

Let V be a 3-dimensional complex manifold with $c_1(V) > 0$. It follows from Yau’s proof of the Calabi conjecture [39] that V has a Kähler metric of positive Ricci curvature. Consequently, V is simply-connected by a theorem of Kobayashi [25] and $H^2(V, \mathbb{C}) = H^{1,1}(V)$ by Bochner’s theorem. It then follows that V is projective, by the Kodaira embedding theorem. The condition $c_1(V) > 0$ is also equivalent to the anticanonical sheaf $\mathcal{O}(-K_V)$ being ample, and any such manifold V is called a *Fano 3-fold*.

For every Fano 3-fold V , a generic divisor D in the anticanonical linear system $| -K_V |$ is a K3 surface, by a theorem of Shokurov [35]. The self-intersection class of $D \subset V$ is Poincaré dual to the restriction $c_1(V)|_D$ and therefore cannot be trivial. However, we have

Proposition 6.42. *Suppose that a Kähler 3-fold V and a surface $D \in |-K_V|$ are given. Let C be a smooth curve in D representing $D \cdot D$ and $\sigma : \tilde{V} \rightarrow V$ the blow-up of V along C .*

Then the closure \tilde{D} of $\sigma^{-1}(D \setminus C)$ is a smooth anticanonical divisor on \tilde{V} and $\tilde{D} \cdot \tilde{D} = 0$. Further, σ restricts to give an isomorphism $\tilde{D} \rightarrow D$ of complex surfaces and this isomorphism identifies the induced Kähler metric on $D \subset V$ with the restriction to \tilde{D} of some Kähler metric on \tilde{V} .

Proof. See e.g. [18], pp.608–609. The formula for the anticanonical class of a blow-up yields, for a curve in a 3-fold, $|-K_{\tilde{V}}| = |\sigma^*(D) - E| = |\tilde{D}|$, where E is the exceptional divisor. It is clear that σ_* identifies the cycle $\tilde{D} \cdot \tilde{D}$ with a cycle on V . But $\sigma_*(\tilde{D} \cdot \tilde{D}) = 0$ (cf. blowing-up a surface at a point). For the claim on Kähler metrics, it can be shown in a standard way, cf. pp.186–187 op.cit., that there is a closed semi-positive $(1, 1)$ -form $\omega_{[E]}$ representing $c_1([E])$ with the following properties: $\omega_{[E]} = 0$ on \tilde{D} and if ω_V is a positive $(1, 1)$ -form on V then $\sigma^*\omega_V - k^{-1}\omega_{[E]}$ is positive on \tilde{V} for any sufficiently large k . \square

Corollary 6.43. *Let V be a Fano 3-fold, $D \in |-K_V|$ a K3 surface, and \tilde{V} the blow-up of V along a curve $D \cdot D$ as in Proposition 6.42. Then $\tilde{V} \setminus \tilde{D}$ has a complete smooth metric with holonomy $SU(3)$, making $\tilde{V} \setminus \tilde{D}$ into a manifold with asymptotically cylindrical end, as defined in Theorem 2.4. Also, $\pi_1(\tilde{V} \setminus \tilde{D}) = 1$.*

Proof. An exceptional curve $\ell = \sigma^{-1}(x)$ in \tilde{V} (where $x \in C$) meets \tilde{D} in exactly one point, $\ell \cdot \tilde{D} = 1$. The corollary now follows from Theorem 2.7 and Proposition 6.42, putting $\overline{W} = \tilde{V}$ (recall that for a blow-up, $\pi_1(\tilde{V}) = \pi_1(V) = 1$). \square

Consider now two pairs $(\tilde{V}_1, \tilde{D}_1), (\tilde{V}_2, \tilde{D}_2)$, where each pair is obtained by blowing up a curve $D_j \cdot D_j$ in a Fano 3-fold V_j and lifting a K3 divisor $D_j \in |-K_{V_j}|$ via the proper transform, as above. Put $W_j = \tilde{V}_j \setminus \tilde{D}_j$. The connected sum construction in §4 for the pair $W_1 \times S^1$ and $W_2 \times S^1$ requires the matching condition to hold for the pair of K3 surfaces $\tilde{D}_j \subset \tilde{V}_j$ and their Kähler classes. In view of Proposition 6.42 and Corollary 6.43, the possibility to satisfy the matching condition for \tilde{D}_1 and \tilde{D}_2 can be decided by considering the divisors D_1 and D_2 on the Fano 3-folds. In general, the Kähler classes induced on D_1 and D_2 by the respective embeddings, may force different volumes of D_1 and D_2 . So it may be necessary to rescale the projective metrics on V_1 or V_2 by a constant factor to achieve the matching K3 divisors.

If the two Kähler K3 surfaces D_j do match, and so a compact real 7-manifold M and a G_2 -structure φ_T on M are well-defined, then $b_1(M) = 0$ by Theorem 4.28(i). Hence M has a metric with holonomy G_2 , by Theorem 5.34. As a short-hand we shall say that the compact G_2 -manifold M is *constructed from the pair of Fano 3-folds V_1 and V_2* . This presumes, as the initial step, appropriate choices of the anticanonical K3 divisors D_1 and D_2 and rescaling of the metrics on V_1 and V_2 if necessary.

Here is the main theorem of this section. By a deformation class we mean a maximal family of smooth complex manifolds which are deformations of each other.

Theorem 6.44. *Let \mathcal{V}_1 and \mathcal{V}_2 be two (not necessarily distinct) deformation classes of Fano 3-folds. There exist representatives $V_j \in \mathcal{V}_j$, $j = 1, 2$, such that a compact G_2 -manifold can be constructed from the pair V_1 and V_2 .*

Remarks. (i) There is a complete classification of Fano 3-folds into 104 deformation classes [21, 32], see also Chapter 12 in [22] for a summary list including the values of basic invariants. This leads to an interesting question of finding the ‘geography’ of the respective G_2 -manifolds, e.g. plotting their topological invariants as in [24], Figure 12.3. The issue is not taken up here, however we do discuss in §8 concrete examples of the *new* topological types of G_2 -manifolds obtained by Theorem 6.44.

(ii) We also point out that it is not absolutely necessary to start from quasiprojective varieties obtained from blow-ups of Fano 3-folds. Indeed, the reader will notice that we really only needed our 3-folds \tilde{V} to be fibred over a Riemann surface with a K3 fibre in the anticanonical class and that the complement of a fibre have finite fundamental group. Examples of G_2 -manifolds will arise from any pair of such fibred spaces provided that a K3 fibre in each space is chosen so that the matching condition is satisfied. It is likely that further examples of matching pairs of K3 fibrations will also be found, e.g. using singular Fano varieties.

As one may expect from the previous arguments, the proof of Theorem 6.44 is largely a matter of finding the matching anticanonical K3 divisors. Before dealing with that, we need to identify the type of K3 surfaces which occur in the anticanonical linear systems on Fano 3-folds in a given deformation class \mathcal{V} .

If V is a Fano 3-fold then any K3 surface D in $|-K_V|$ is necessarily projective-algebraic. Given a projective embedding of V , it is clear that the degree of an algebraic K3 surface in the anticanonical linear system is determined by the deformation class of Fano 3-fold, but in fact more is true. By Lefschetz hyperplane theorem, in the form due to Bott [7], the embedding $\iota : D \hookrightarrow V$ induces an injective homomorphism $\iota^* : H^2(V, \mathbb{Z}) \rightarrow H^2(D, \mathbb{Z})$ with $H^2(D, \mathbb{Z})/\iota^*H^2(V, \mathbb{Z})$ torsion-free. The latter property follows by standard topology as both D and V are simply-connected and up to a homotopy equivalence V is obtained from D by attaching cells of real dimension at least 3. Similarly, the Kähler class of V defined by a projective embedding is primitive (indivisible by an integer > 1) and the induced Kähler class of D is primitive in $H^2(D, \mathbb{Z})$. Recall that a sublattice is called primitive when the quotient of this sublattice has no torsion. Thus for every K3 anticanonical divisor D on a Fano 3-fold V in a given deformation class \mathcal{V} , the Picard lattice $H^2(D, \mathbb{Z}) \cap H^{1,1}(D)$ necessarily contains a primitive sublattice $S(\mathcal{V}) = \iota^*H^2(V, \mathbb{Z})$. We use the notation $S(\mathcal{V})$ to indicate that this sublattice is determined by the deformation class of V . (We remind that the K3 surfaces in $|-K_V|$ form a Zariski open, hence connected, family.)

The algebraic K3 surfaces whose Picard lattice contains a primitively embedded non-degenerate lattice S are studied in [14] and called ‘ample S -polarized K3 surfaces’ if S contains a Kähler class. The cup-product on Picard lattice of any algebraic surface has positive index exactly 1, by Hodge index theorem. As a sublattice of the form $S = S(\mathcal{V})$ contains an ample class, it is non-degenerate of signature $(1, t)$, where the rank $1+t = b^2(V)$ is equal to the Betti number of some (any) $V \in \mathcal{V}$. By the classification result, any Fano

3-fold has $1 \leq b^2(V) \leq 10$ so the possible values of t are $0 \leq t \leq 9$. Thus the inclusion map defines an ample $S(\mathcal{V})$ -polarization for every anticanonical K3 divisor on $V \in \mathcal{V}$.

Recall that the integral second cohomology of a K3 surface considered with the cup-product is isometric to the (unique) even unimodular lattice L of signature $(3, 19)$. In fact, $L = (-E_8) \oplus (-E_8) \oplus H \oplus H \oplus H$, where $H = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ denotes the hyperbolic plane lattice and E_8 is a standard (even and positive-definite) root-lattice. The constraint on the signature of $S(\mathcal{V})$ found above implies that a primitive embedding of any $S(\mathcal{V})$ into L is unique up to an isometry of L , by [14] Corollary 5.2 (see also [33] Theorem 1.14.4). A map $f : D \rightarrow D'$ is defined to be an isomorphism of $S(\mathcal{V})$ -polarized K3 surfaces if f is an isomorphism of complex K3 surfaces and f^* intertwines the embeddings of $S(\mathcal{V})$ in the respective Picard lattices. In combination with the global Torelli theorem this tells us that the class of $S(\mathcal{V})$ -polarized K3 surfaces is determined by the deformation class \mathcal{V} of Fano 3-folds. We shall denote the moduli space of all the isomorphism classes of $S(\mathcal{V})$ -polarized K3 surfaces by $\mathcal{K}\mathcal{Z}(\mathcal{V})$.

When $b^2(V) = 1$, the lattice $S(\mathcal{V})$ is generated by one element, an integral (Kähler) class on D of positive square say $2n - 2$ ($n \geq 2$). In this case $\mathcal{K}\mathcal{Z}(\mathcal{V}) = \mathcal{K}\mathcal{Z}(n)$ is just the familiar 19-dimensional moduli space of algebraic K3 surfaces of degree $2n - 2$ in $\mathbb{C}P^n$. The general case is quite similar. Firstly, by application of the embedding result of Nikulin ([33], Theorem 1.12.4), any lattice $S(\mathcal{V}_1)$ admits a primitive embedding into the even unimodular lattice $L_0 = (-E_8) \oplus (-E_8) \oplus H \oplus H$ of signature $(2, 18)$, because $S(\mathcal{V}) \otimes \mathbb{R}$ embeds in $L_0 \otimes \mathbb{R}$ and $\text{rk } S(\mathcal{V}) \leq \frac{1}{2} \text{rk } L_0$. Therefore, the orthogonal complement to $S(\mathcal{V}_1)$ in the K3 lattice L contains a copy of H . It then follows from [14] Theorem 5.6 that the moduli space $\mathcal{K}\mathcal{Z}(\mathcal{V})$ is an irreducible (hence connected) quasiprojective algebraic variety of (complex) dimension $20 - b^2(V)$.

We shall need the following result, which may be viewed as a kind of converse to Shokurov's theorem cited earlier.

Theorem 6.45. *Let \mathcal{V} be a deformation class of Fano 3-folds. Denote by $\mathcal{P}(\mathcal{V})$ the space of all pairs (V, D) for $V \in \mathcal{V}$ and a K3 surface $D \in |-K_V|$. Then the image of the forgetful map*

$$(6.46) \quad \pi : (V, D) \in \mathcal{P}(\mathcal{V}) \rightarrow D \in \mathcal{K}\mathcal{Z}(\mathcal{V})$$

is Zariski open (in particular, dense) in $\mathcal{K}\mathcal{Z}(\mathcal{V})$.

Remark. As will be discussed in the next sections, for a number of deformation classes \mathcal{V} , specific properties of 3-folds in \mathcal{V} ensure that the map π is actually surjective. We do not know if π is surjective for an arbitrary deformation class \mathcal{V} of Fano 3-folds.

Theorem 6.45 is proved in the next section.

Proof of Theorem 6.44 assuming Theorem 6.45. With the above preparations, the proof will be accomplished almost entirely via constructions on the K3 lattice L . (The reader unaccustomed to the K3 lattice may prefer to assume in the first reading that the Fano 3-folds in $\mathcal{V}_1, \mathcal{V}_2$ have Betti number $b^2 = 1$. This would simplify certain technical issues

of lattice arithmetics below and make the principal argument more transparent while still applicable to some examples.)

Let D be a K3 surface in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)$ and fix an isometry between $H^2(D, \mathbb{Z})$ and L . This makes $S(\mathcal{V}_1)$ into a primitive sublattice of L of rank $b^2(V_1)$, $V_1 \in \mathcal{V}_1$. Denote by $[\kappa_1]$ the image of the induced Kähler class of $D \subset V_1$, so $[\kappa_1] \in S(\mathcal{V}_1) \subset L$ is a primitive vector of positive square $2n_1 - 2$ say ($n_1 \geq 2$). By the global Torelli theorem, the complex structure on D is determined by the image of the complex line $H^{2,0}(D) \subset H^2(D, \mathbb{C})$ in $L \otimes \mathbb{C}$ under the isometry. The space $H^{2,0}(D)$ is equivalent to $P(D) = H^2(D, \mathbb{R}) \cap (H^{2,0} \oplus H^{0,2})(D)$, an oriented positive-definite real 2-plane in $H^2(D, \mathbb{R})$. For a K3 surface D in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)$, the plane $P(D)$ is necessarily orthogonal to the sublattice $S(\mathcal{V}_1)$ of the Picard lattice of D . Conversely, a standard result for the K3 surfaces, known as surjectivity of the period map ([5] §VIII.14), implies that *every* positive 2-plane P in $L \otimes \mathbb{R}$ orthogonal to $S(\mathcal{V}_1)$ occurs as $P(D)$ for some $D \in \mathcal{K}\mathcal{Z}(\mathcal{V}_1)$.

As was explained above, the lattice $S(\mathcal{V}_1)^\perp$ contains H and therefore for any given $n_2 \geq 2$, we can find a primitive vector $[\kappa_2] \in S(\mathcal{V}_1)^\perp$ of positive length $([\kappa_2], [\kappa_2]) = 2n_2 - 2$. In particular, we may take (we shall) $2n_2 - 2$ to be the square of the Kähler class of K3 surfaces in $\mathcal{K}\mathcal{Z}(\mathcal{V}_2)$. It is not difficult to check that $[\kappa_1], [\kappa_2]$ can be extended to a basis of L , hence these two vectors generate a primitive sublattice of L (this will become important soon).

If the map π of Theorem 6.45 is not surjective for $\mathcal{V} = \mathcal{V}_1$ then one more step in the construction is needed. We shall say that a K3 surface $D \in \mathcal{K}\mathcal{Z}(\mathcal{V}_1)$ is \mathcal{V}_1 -*generic* (or just generic when a confusion is unlikely) if D is in the image of the projection π , $D \in \pi(\mathcal{P}(\mathcal{V}_1))$. Now the family of positive 2-planes P through $[\kappa_2]$ and orthogonal to $S(\mathcal{V}_1)$ defines a real-analytic subvariety $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ of *real* dimension $20 - b^2(V_1)$ in the moduli space $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)$ of complex dimension $20 - b^2(V_1)$. The real subvariety $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ is locally modelled on the projective space P_1^+ of real lines in the positive cone of $L_1 \otimes \mathbb{R}$, where $L_1 = (S(\mathcal{V}_1) \oplus \langle [\kappa_2] \rangle)^\perp$ is a Lorentzian sublattice defined by taking the orthogonal complement in L . On the other hand, local Torelli theorem implies that the complex moduli space $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)$ is locally biholomorphic to the period domain $\{\omega \mathbb{C} \in \mathbb{P}(S(\mathcal{V}_1)^\perp \otimes \mathbb{C}) : (\omega, \omega) = 0, (\omega, \bar{\omega}) > 0\}$. We deduce by examination of the tangent spaces that $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ is *not* contained in any complex subvariety of positive codimension in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)$. Thus $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ cannot lie in any Zariski closed subset of $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)$ and has to contain points defined by generic K3 surfaces. The complement of these generic period points in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ is the common zero set of a finite system of real-analytic functions. By the identity theorem, this analytic subset cannot contain open neighbourhoods, so the points in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ defined by generic K3 surfaces form a dense open subset.

Now everything in the above constructions concerning $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)$ can be repeated for $\mathcal{K}\mathcal{Z}(\mathcal{V}_2)$, with the obvious change of notation. We then obtain a sublattice $S(\mathcal{V}_2)$ of L containing a primitive vector $[\kappa'_2]$ of length $2n_2 - 2$, corresponding to the Kähler class of a generic projective surface in $\mathcal{K}\mathcal{Z}(\mathcal{V}_2)$. We also obtain a primitive vector $[\kappa'_1] \in L$ of positive length $2n_1 - 2$ orthogonal to $S(\mathcal{V}_2)$ and such that $[\kappa'_1]$ and $[\kappa'_2]$ generate a primitive rank 2 positive-definite sublattice of L isometric to the previously obtained sublattice generated by $[\kappa_1]$ and $[\kappa_2]$. By [5], Theorem I.2.9(ii), any two primitive embeddings of a rank 2 even

lattice into L can be intertwined by an isometry of L , so we may assume without loss of generality that $[\kappa'_1] = [\kappa_1]$ and $[\kappa'_2] = [\kappa_2]$.

For the final step in the proof, consider first the special case when the Fano 3-folds in \mathcal{V}_j , $j = 1, 2$, have $b^2 = 1$, so the sublattice $S(\mathcal{V}_j)$ is generated by $[\kappa_j]$. Then, by the construction and by the surjectivity of the period map, any vector, say $[\kappa_K] \in L \otimes \mathbb{R}$, of positive length and orthogonal to both $[\kappa_1]$ and $[\kappa_2]$ simultaneously gives us *two* K3 surfaces, say $D_1 \in \mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ and $D_2 \in \mathcal{K}\mathcal{Z}(\mathcal{V}_2)^\mathbb{R}$. The positive 2-plane $P(D_1)$ is spanned by $[\kappa_2]$ and $[\kappa_K]$, whereas $P(D_2)$ is spanned by $[\kappa_1]$ and $[\kappa_K]$. Furthermore, a vector $[\kappa_K]$ can be chosen so that the two K3 surfaces are generic, as this just amounts to choosing an element $[\kappa_K]\mathbb{R}$ in the intersection of the two dense open subsets of P_1^+, P_2^+ identified above, as in this case $P_1^+ = P_2^+ \subseteq \mathbb{P}(([\kappa_1]\mathbb{R} \oplus [\kappa_2]\mathbb{R})^\perp) \subset \mathbb{P}(L \otimes \mathbb{R})$.

It remains to multiply the classes $[\kappa_i]$ by constants to define $[\kappa_I], [\kappa_J]$ such that $[\kappa_I]^2 = [\kappa_J]^2 = [\kappa_K]^2$. Note that this rescaling changes the Kähler classes but does not change the complex structures of D_1 and D_2 . Thus we have constructed D_1 and D_2 satisfying the matching condition, as defined in §4. These K3 surfaces are generic and occur as anti-canonical divisors in Fano 3-folds V_1, V_2 in $\mathcal{V}_1, \mathcal{V}_2$ respectively, by Theorem 6.45. Therefore, and in view of the remarks earlier in this section, the construction of a G_2 -manifold from V_1 and V_2 can go ahead. This completes the proof for the Fano 3-folds having $b^2 = 1$.

In general, when Fano 3-folds may have $b^2 \geq 1$, the choice of a suitable $[\kappa_K]$ follows the same idea but is slightly technical. We shall need an auxiliary lemma. Recall from the above the definitions of L_1 and P_1^+ .

Lemma 6.47. *Let U_1 be a dense open set in P_1^+ and C^+ the image of the positive cone in $\mathbb{P}(S \otimes \mathbb{R})$, where $S \subset L_1$ is a non-degenerate primitive sublattice of signature $(1, 1)$. Then there exists an isometry of L fixing $S(\mathcal{V}_1) \cup [\kappa_2]$ and mapping an element of U_1 into C^+ .*

Proof. We assume that $C^+ \cap U_1$ is empty since otherwise there is nothing to prove. As the lattice S is indefinite it contains an isotropic vector x which may be chosen primitive. Let x' be such that $\{x, x'\}$ is a basis of S . An open set U_1 contains a rationally defined element $e\mathbb{R}$, with $e \in L_1$ of positive length. Multiplying e by an integer if necessary, we may write $e = ax + bx' + (x, x')bf$, $a, b \in \mathbb{Z}$, for some $f \in L_1$, $f \neq 0$, orthogonal to S . As U_1 is dense open, we may assume $b \neq 0$ in the latter formula. Then Eichler's 'elementary transformation' with respect to f and x ,

$$\varepsilon_{f,x} : v \in L \rightarrow v + (v, f)x - \frac{1}{2}(f, f)(v, x)x - (v, x)f \in L,$$

provides the required isometry of L , as it maps e to the span of x and x' . As x and f are both orthogonal to $S(\mathcal{V}_1) \cup [\kappa_2]$, the latter set is fixed by $\varepsilon_{f,x}$. \square

Recall that we have two different views on the K3 lattice L (and on the vectors $[\kappa_1], [\kappa_2]$ in L) depending on whether we are considering K3 surfaces in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ or in $\mathcal{K}\mathcal{Z}(\mathcal{V}_2)^\mathbb{R}$. The sublattices $S(\mathcal{V}_1), S(\mathcal{V}_2)$ together generate a non-degenerate sublattice of L of signature $(2, t)$, $t \leq 18$, so the orthogonal complement in L of this sublattice has rank at least 2 and contains a non-degenerate sublattice S of signature $(1, 1)$. Take up one point of view on L . Applying, if necessary, Lemma 6.47 to S , with U_1 the subset of generic points in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$, we may assume that there is a vector $[\kappa_K] \in S \otimes \mathbb{R}$ defining a generic K3

surface D_1 in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R} \subset \mathcal{K}\mathcal{Z}(\mathcal{V}_1)$. Moreover the set of the ‘bad’ elements $[\kappa_K]\mathbb{R} \in C^+$ giving non-generic K3 surfaces in $\mathcal{K}\mathcal{Z}(\mathcal{V}_1)^\mathbb{R}$ is an analytic subset, hence discrete in the 1-dimensional C^+ . Note that we are looking at the structure $(S(\mathcal{V}_1) \cup [\kappa_2]) \subset L$, but the sublattice $S(\mathcal{V}_2)$ is *disregarded* for the moment (except for the vector $[\kappa_2]$), i.e. the action of the isometry given in Lemma 6.47 is not applied to $S(\mathcal{V}_2)$. Now shifting to the other view of L we consider $S(\mathcal{V}_2)$ (and temporarily disregarding $S(\mathcal{V}_1)$ except for the $[\kappa_1]$) we apply, if necessary, Lemma 6.47 again to the same S but now with U_2 , P_2^+ , L_2 , respectively defined using the sublattice $S(\mathcal{V}_2)$. This similarly gives us another dense open subset in C^+ , consisting of those $[\kappa_K]\mathbb{R}$ in the positive cone of $S \otimes \mathbb{R}$ which define \mathcal{V}_2 -generic K3 surfaces. Intersecting the two dense open subsets of C^+ we get a choice of $[\kappa_K]$ that we want. This completes the proof of Theorem 6.44. \square

7. INFINITESIMAL DEFORMATIONS OF THE ANTICANONICAL K3 DIVISORS

In this section we prove Theorem 6.45: that every K3 surface D in a Zariski open subset in the moduli space $\mathcal{K}\mathcal{Z}(\mathcal{V})$ of lattice-polarized K3 surfaces occurs as an anticanonical divisor in some Fano 3-fold $V \in \mathcal{V}$. In the case, when the Fano 3-folds in \mathcal{V} have $b^2 = 1$ (and so $\mathcal{K}\mathcal{Z}(n)$ is the moduli space of all the projective K3 surfaces of appropriate degree) this is a known result, proved in [13] by a different method. The results of that work, in particular, give a list of (deformation classes of) those Fano 3-folds with $b^2 = 1$ for which *every* K3 surface (of appropriate degree) occurs as an anticanonical divisor on some member of \mathcal{V} , i.e. in our notation, the projection π is then surjective.

The argument given below does not use any assumption on Betti numbers or consideration of the list of Fano 3-folds. The basic idea is that the space of pairs $\mathcal{P}(\mathcal{V})$ can be parametrized by a Zariski open set in a (complex) projective variety, similarly to (an irreducible component of) Chow parameter space for all the smooth varieties of a given degree in a given projective space. Cf. §5.4 in [12]. Then π may be thought of as a regular map between quasiprojective varieties. By a theorem of Chevalley, [19] p.94, the image of π is then a ‘constructible subset’ of $\mathcal{K}\mathcal{Z}(n)$, which means that it is a finite disjoint union of sets, each one being the intersection of a Zariski open and a Zariski closed subset of $\mathcal{K}\mathcal{Z}(n)$. We have checked that $\mathcal{K}\mathcal{Z}(\mathcal{V})$ is connected, so it now suffices to show that the image of π has the right dimension to obtain that $\pi(\mathcal{P}(\mathcal{V}))$ is Zariski open in $\mathcal{K}\mathcal{Z}(\mathcal{V})$, as claimed.

Proposition 7.48. *The dimension of $\pi(\mathcal{P}(\mathcal{V}))$ is equal to the dimension $20 - b^2(V)$ of $\mathcal{K}\mathcal{Z}(V)$.*

Proof. We shall find the dimension by examining the action of π on the local first order infinitesimal deformations of a pair (V, D) in $\mathcal{P}(\mathcal{V})$. The argument relies on the familiar Kodaira–Spencer–Kuranishi theory of deformations of the holomorphic structures [26], see also §2 in [15] for a related discussion and further references. We recall, in outline, that the isomorphism classes of the deformations of a compact complex manifold D , respectively V , define elements in Čech cohomology group H^1T_D , H^1T_V , where T_D, T_V denote the sheaves of holomorphic local vector fields. On the other hand, any infinitesimal deformation in H^1T_V, H^1T_D extends to a genuine deformation of the complex manifold if the obstruction

space H^2T_V , H^2T_D , vanishes. This is the case in the present situation as, using Serre duality and the Kodaira vanishing theorem ([18], Ch.1), we have $H^2T_V = H^1(\Omega_V^1(-D)) = 0$ for a negative line bundle $[-D]$ on a Fano 3-fold and $H^2T_D = H^0\Omega_D^1 = H^{0,1}(D) = 0$ on a K3 surface. Notice also that similarly $h^1T_D = h^{1,1}(D) = 20$ equals the dimension of the moduli space $\mathcal{K}\mathcal{S}$ of all complex K3 surfaces and $H^0T_D = H^{2,1}(D) = 0$ verifies that the group of automorphisms of a K3 surface is discrete.

A suitable way to deal with deformations of divisors coming from deformations of the ambient manifold is discussed in §1 of [38]. We briefly review the technique adapting it to our situation when the line bundle is determined by the complex manifold V . To deal with the pairs (V, D) , consider the morphism (more formally, the complex) of sheaves

$$d_s : T_V \oplus T_D \rightarrow T_V|_D,$$

where d_s is the sum of the morphisms appearing in the tangential exact sequence

$$(7.49) \quad 0 \rightarrow T_D \rightarrow T_V|_D \rightarrow N_D \rightarrow 0$$

and in the exact sequence

$$(7.50) \quad 0 \rightarrow T_V(-D) \rightarrow T_V \rightarrow T_V|_D \rightarrow 0,$$

obtained by tensoring the structure sequence of D with T_V . Associated to d_s is the double complex of Čech cochains with respect to an (appropriately refined) open covering \mathcal{U} of V ,

$$\begin{array}{ccccc} C^0(\mathcal{U}, T_V \oplus \iota_* T_D) & \longrightarrow & C^1(\mathcal{U}, T_V \oplus \iota_* T_D) & \longrightarrow & \dots \\ \downarrow d_s & & \downarrow -d_s & & \\ C^0(\mathcal{U}, T_V|_D) & \longrightarrow & C^1(\mathcal{U}, T_V|_D) & \longrightarrow & \dots \end{array}$$

The homology of the corresponding single complex is called the hypercohomology \mathbb{H}^* of d_s . The hypercohomology has the same meaning for the deformations of pairs (V, D) as the cohomology of tangent sheaves does for the deformations of complex manifolds. In particular, the isomorphism classes of linear infinitesimal deformations of the pair (V, D) are canonically parametrized by the first hypercohomology group \mathbb{H}^1 of d_s (cf. Proposition 1.2 in [38]).

The spectral sequence of hypercohomology (see [18] §3.5) of d_s yields an exact sequence

$$0 \rightarrow \mathbb{H}^1 \xrightarrow{\beta} H^1T_V \oplus H^1T_D \xrightarrow{d_s} H^1(T_V|_D) \rightarrow \mathbb{H}^2 \rightarrow 0,$$

where we took account of the vanishing of H^2T_V and H^2T_D . The monomorphism β is the ‘infinitesimal embedding’ of the deformations of the pair (V, D) into the pairs of deformations of V and D alone. We are actually interested in the composition $\beta_2 : \mathbb{H}^1 \rightarrow H^1T_D$ of β with the forgetful projection to H^1T_D .

To understand β_2 , first take the cohomology of (7.49) to obtain an exact sequence

$$(7.51) \quad H^0N_D \rightarrow H^1T_D \rightarrow H^1(T_V|_D) \rightarrow 0.$$

Here we used the isomorphism $N_D = \mathcal{O}_V(D)|_D$ (the adjunction formula), then Serre duality on D and the Kodaira vanishing to claim $H^1N_D = H^1(\mathcal{O}_V(D)|_D) = H^1(\mathcal{O}_V(-D)|_D) = 0$. Then (7.51) tells us that the map $H^1T_D \rightarrow H^1(T_V|_D)$ is surjective with the kernel given

by those infinitesimal deformations of D which are obtainable by moving D in $|-K_V|$. It follows that d_s is surjective and the obstruction space \mathbb{H}^2 vanishes, hence *every* class in \mathbb{H}^1 arises from a genuine deformation of the pair (V, D) and so the image of β_2 coincides with the image of the infinitesimal action of π on (V, D) .

Finally, taking the cohomology of (7.50) we can write an exact sequence

$$H^1 T_V \rightarrow H^1(T_V|_D) \xrightarrow{\gamma} H^2(T_V(-D)) \rightarrow 0$$

(remember $H^2 T_V = 0$). Using Serre duality we write $H^2(T_V(-D)) = H^1 \Omega_V^1 = H^{1,1}(V) = H^2(V, \mathbb{C})$. As γ is surjective, the image of $H^1 T_V$ has (complex) codimension $b_2 = b_2(V)$ in $H^1(T_V|_D)$. Then, putting all the above conclusions together, we calculate that the dimension of the image of β_2 is precisely $20 - b_2(V)$. \square

8. NEW TOPOLOGICAL TYPES OF COMPACT G_2 -MANIFOLDS

We next look at the topology of the compact G_2 -manifolds constructed from pairs of Fano 3-folds. The notation is as set up in §6.

Corollary 6.43 and Theorem 4.28(i) imply that any G_2 -manifold constructed from two Fano 3-folds is simply-connected. We then consider the integral homology groups. As H_1 is determined by the fundamental group and in view of Poincaré duality, we restrict attention to H_2 and H_3 . Much of the calculation is the linear algebra of Mayer–Vietoris exact sequences and is fairly straightforward.

Consider first the blow-up \tilde{V} of a Fano 3-fold V along a curve $C = D \cdot D$. The genus of C is usually referred to as the genus $g(V)$ of Fano 3-fold and is calculated by the formula $g(V) = -K_V^3/2 + 1 = \langle c_1(V)^3, [V] \rangle / 2 + 1$ (e.g. [21], Proposition 1.6). Note that $-K_V^3$ is necessarily positive on a Fano 3-fold V , in particular $g(V) \geq 2$. Now the exceptional divisor on \tilde{V} is a projective bundle of the Riemann spheres over C and applying standard results on the cohomology of a blow-up ([18], p.605) we find that

$$(8.52) \quad H_2(\tilde{V}) \cong H_2(V) \oplus \mathbb{Z} \quad \text{and} \quad H_3(\tilde{V}) \cong H_3(V) \oplus \mathbb{Z}^{2g(V)}.$$

We have already calculated the effect of removing a K3 divisor D from $\tilde{V} = \overline{W}$ on the fundamental group, in the course of proving Theorem 2.7. Further application of Mayer–Vietoris theorem to $\overline{W} = U \cup W$, with a neighbourhood U contracting to D , yields

$$(8.53) \quad H_2(\overline{W}) = H_2(W) \oplus \mathbb{Z} \quad \text{and} \quad H_3(W) = H_3(\overline{W}) \oplus \mathbb{Z}^{22-b_2(V)}$$

(as $b_2(V) = b_4(\overline{W}) - 1$). For the second isomorphism in (8.53), recall first that $H_i(Y \times S^1) = H_i(Y) \oplus H_{i-1}(Y)$ for any manifold Y . Then note that the boundary homomorphism $H_4(\overline{W}) \rightarrow H_3(D \times S^1) = H_2(D)$ evaluates the intersection cycle with D . The kernel is generated by the cycle of D in $H_4(\overline{W})$, because $D \cdot D = 0$ in \overline{W} , but if a divisor D' is not in the anticanonical class then $D' \cdot D \neq 0$ (this can be calculated on V , away from the blow-up locus C).

Now for the compact real 7-manifold M . Up to a homotopy equivalence,

$$M \sim (W_1 \times S^1) \cup (W_2 \times S^1), \quad \text{such that} \quad (W_1 \times S^1) \cap (W_2 \times S^1) \sim D \times S^1 \times S^1.$$

From Mayer–Vietoris theorem and the knowledge of the homology of K3 surface we can obtain an exact sequence

$$H_2(D) \oplus \mathbb{Z} \xrightarrow{\gamma_1} H_2(W_1) \oplus H_2(W_2) \rightarrow H_2(M) \rightarrow 0,$$

as $H_1(W_i)$, $i = 1, 2$, is trivial. Hence

$$(8.54) \quad H_2(M) = \frac{H_2(W_1) \oplus H_2(W_2)}{\gamma_1(H_2(D))}.$$

To understand the homomorphism γ_1 recall that each of the two push-forward homomorphisms $H_2(D) \rightarrow H_2(V_j)$ is surjective by Lefschetz hyperplane theorem. From the way we identified the matching K3 surfaces in the construction of M we can further deduce that the image of γ_1 contains non-zero elements $(0, \gamma_1(\text{P.D.}[\kappa_1]))$ and $(\gamma_1(\text{P.D.}[\kappa_2]), 0)$. The group $H_2(M)$ will have no torsion if both $H_2(W_i)$ are torsion-free. (For the latter, e.g. $H^3(V_i) = 0$ will suffice.) The dimension of $\gamma_1(H_2(D))$, and hence the Betti number $b_2(M)$, is not in general uniquely determined by V_1, V_2 because it depends on the rank of the sublattice of $H^2(D)$ generated by $S(\mathcal{V}_1)$ and $S(\mathcal{V}_2)$. In any event, we have

$$(8.55) \quad b_2(M) \leq \min\{b_2(V_1), b_2(V_2)\} - 1.$$

On the other hand, we shall see an example below when all the values of $b_2(M)$ allowed by (8.55) actually occur. As any Fano 3-fold V_j has $1 \leq b_2(V_j) \leq 10$, any G_2 -manifold M constructed from a pair of Fano 3-folds necessarily satisfies

$$0 \leq b_2(M) \leq 9.$$

The group $H_3(M)$ is found by similar, although slightly lengthy computations, taking account of the previous steps. We omit the details. Note, in particular, that the inclusion homomorphism of $H_3(D \times S^1 \times S^1)$ is equivalent to the direct sum of two maps $H_2(D) \rightarrow H_2(W_{3-i}) \oplus H_3(W_i)$, $i = 1, 2$, and these maps miss the subspaces $H_3(\overline{W}_i) \subset H_3(W_i)$. The dimension counting once again depends on the rank of the sublattice generated by $S(\mathcal{V}_1)$ and $S(\mathcal{V}_2)$ and yields (substituting $\tilde{V}_j = \overline{W}_j$)

$$(8.56) \quad b_3(M) + b_2(M) = b_3(\tilde{V}_1) + b_3(\tilde{V}_2) + 23,$$

for any G_2 -manifold M constructed from V_1 and V_2 .

In the case when one of the two Fano 3-folds has $b_2 = 1$ we can say more.

Theorem 8.57. *Suppose that M is a compact 7-manifold with holonomy G_2 constructed from Fano 3-folds V_1 and V_2 , where $b^2(V_1) = 1$. Then*

$$b_2(M) = 0 \quad \text{and} \quad b_3(M) = b_3(V_1) - K_{V_1}^3 + b_3(V_2) - K_{V_2}^3 + 27.$$

Further, the diffeomorphism type of M (as a smooth real 7-manifold) is independent of the choice of V_1 and V_2 in their deformation classes.

Proof. The assertions, except for the last sentence, are immediate from the above topological considerations. The diffeomorphism type of a smooth real 6-manifold underlying W is not affected by moving an anticanonical K3 divisor or deforming the complex structure of a Fano 3-fold. Thus the only possible ambiguity in defining a smooth 7-manifold M is

in the choice of the matching diffeomorphism of the K3 divisors (recall §4). Let a piece of M , diffeomorphic to $W_1 \times S^1$, be fixed and consider two choices of matching K3 surfaces $D_1 \rightarrow D_2$ and $D_1 \rightarrow D'_2$. Respectively, there are two choices $W_2 \times S^1$ or $W'_2 \times S^1$ for attaching the other piece; denote the resulting 7-manifolds by M and M' . It then remains to demonstrate that the identity map of $W_1 \times S^1$ can be extended to a diffeomorphism $M \rightarrow M'$. The $\text{id}_{W_1 \times S^1}$ readily extends to the diffeomorphism between the cylindrical ends $D \times S^1 \times S^1 \times \mathbb{R}$ of $W_2 \times S^1$ and $W'_2 \times S^1$ which is determined by a diffeomorphism of the real 4-manifold D underlying K3 surfaces D_2, D'_2 . We claim that this latter diffeomorphism is isotopic (can be connected by a smooth path of automorphisms) to the identity map of D . Assuming this claim, it is easy to see that M is diffeomorphic to M' . To verify the claim, we refer back to the lattice considerations and the notation in the proof of Theorem 6.44. A choice of D_2 or D'_2 corresponds, in the first place, to a choice of positive lattice vector $[\kappa_2]$, orthogonal to a given $[\kappa_1]$, but all the choices of $[\kappa_2]$ are equivalent up to an isometry of the K3 lattice L . (It is at this point that the condition $b^2(V_1) = 1$ is used.) Then the ambiguity comes down to a choice of positive $[\kappa_K] \in L \otimes \mathbb{R}$. But any two such choices are connected by a path of positive vectors. Then Torelli theorem and surjectivity of the period map translate this into a path of complex structures from D_2 to D'_2 and induce the required path of the diffeomorphisms of the underlying real manifold D . \square

The G_2 -manifolds with vanishing or finite second homology group give a new series. Previously, only one topological type of G_2 -manifold with $b_2 = 0$ was known (it has $b_3 = 215$), obtained by resolving singularities of the quotient of 7-torus by \mathbb{Z}_2^2 , see [24], §12.5.5.

Here are examples of new G_2 -manifolds constructed from some well-known Fano manifolds and treated rather more explicitly than in the general results in the previous sections.

Example 8.58 (Rigid Fano 3-folds). $V = \mathbb{CP}^3$ is a Fano 3-fold and any smooth quartic is a K3 surface in the anticanonical class, so in this example π maps $\mathcal{P}(\mathbb{CP}^3)$ onto $\mathcal{K}\mathcal{Z}(3)$. More generally, we remark that in the case when the deformation class \mathcal{V} consists of V alone the image of the projection $\pi : \mathcal{P}(\mathcal{V}) \rightarrow \mathcal{K}\mathcal{Z}(\mathcal{V})$ is a *closed* subset in $\mathcal{K}\mathcal{Z}(\mathcal{V})$ and hence π is surjective. The closedness is not difficult to see, for in this case $\mathcal{P}(\mathcal{V})$ is identified with a Zariski open subset in the projective space parametrizing the anticanonical linear system $|-K_V|$ and π becomes a tautological map to the isomorphism class of a divisor.

By Theorem 6.44, a compact G_2 -manifold, say M_0 , can be constructed from two copies of \mathbb{CP}^3 , taking a pair of matching quartics D_1, D_2 . As the cycle of any $D = D_i$ is 4 times the hyperplane cycle, then $-K_V^3 = \frac{1}{2}D \cdot D \cdot D = 64$. The resulting G_2 -manifold has $H_2(M_0) = 0$ by (8.54), and $b_3(M_0) = 155$ by Theorem 8.57.

A Fano 3-fold $V' = \mathbb{CP}^2 \times \mathbb{CP}^1$ has $H^2(V') = \mathbb{Z} \oplus \mathbb{Z}$, so in this case we encounter anticanonical divisors with a rank 2 polarization lattice $S(V')$. The anticanonical divisors on V' are defined by the polyhomogeneous polynomials of bidegree $(3, 2)$ in the respective homogeneous variables. Calculation in the cohomology ring gives $-K_{V'}^3 = 54$. Taking the intersections of 2-cycles defined on $D' \in |-K_{V'}|$ by $\mathbb{CP}^2 \times \text{pt}$ and $\ell \times \mathbb{CP}^1$, where ℓ denotes

a projective line in \mathbb{CP}^2 , we find

$$S(V') = \begin{pmatrix} 0 & 3 \\ 3 & 2 \end{pmatrix}.$$

The induced Kähler class on D is Poincaré dual to the cycle expressed by the vector $(1; 1)$, with respect to the basis of $S(V')$. So D' may be realized as an $S(V')$ -polarized octic K3 surface, indeed it embeds in \mathbb{CP}^5 by Segre embedding of V' . It follows that a G_2 -manifold M' constructed from \mathbb{CP}^3 and $\mathbb{CP}^2 \times \mathbb{CP}^1$ has $H_2(M') = 0$ (the vanishing of $H^3(\mathbb{CP}^3)$ and $H^3(\mathbb{CP}^2 \times \mathbb{CP}^1)$ implies that there is no torsion), and $b_3(M') = 145$, by Theorem 8.57.

For the construction of a G_2 -manifold from two copies of $\mathbb{CP}^2 \times \mathbb{CP}^1$ one needs to take account of the rank of the span of two copies of the polarization lattice $S(V')$ embedded in the K3 lattice (recall the proof of Theorem 6.44). There are two possibilities to consider, as the rank can be 3 or 4.

It is convenient to use a basis $\{f_1 = (1; 1), f_2 = (1; 0)\}$ of $S(V')$ to keep track of the Kähler class f_1 in the embeddings. The intersection numbers in the new basis are $(f_1, f_1) = 8$, $(f_1, f_2) = 3$, $(f_2, f_2) = 0$. Let $\{e_j, e'_j\}$, $j = 1, 2, 3$, be the standard basis of the j -th hyperbolic plane summand H in the K3 lattice, so $(e_j, e_j) = (e'_j, e'_j) = 0$ and $(e_j, e'_j) = 1$. The primitive embedding of two copies of $S(V')$ defined by $f_1 \rightarrow [\kappa_1] = e_1 + 4e'_1$, and $f_1 \rightarrow [\kappa_2] = e_2 + 4e'_2$, with $f_2 \rightarrow e_1 - e'_1 + e_2 - e'_2 + e_3 + 2e'_3$ for both copies, have the property that the span of the two images of $S(V')$ has rank 3. Denote by M_1 the corresponding G_2 -manifold. Then $\pi_1(M_1) = 1$, $H_2(M_1) = \mathbb{Z}$, and $b_3(M_1) = 134$. The previously published examples of G_2 -manifolds with $b_2 = 1$, constructed by a different method, have either $b_3 = 142$ or $b_3 = 186$ ([24] §§12.5.5 and 12.7.4), so the G_2 -manifold M_1 is new.

It is also possible to embed two copies of $S(V')$ in the K3 lattice so that together they generate a lattice of rank 4. This constructs a G_2 -manifold M_2 which is again simply-connected, but now $H_2(M_2)$ vanishes and $b_3(M_2) = 135$. Details are left to the reader.

Example 8.59 (Branched double covers). One such Fano 3-fold is the double cover X_2 of \mathbb{CP}^3 branched over a smooth quartic surface D . For a double cover over a divisor, $b_2(X_2) = b_2(\mathbb{CP}^3)$ (note that by Lefschetz–Bott any 2-cycle on a 3-fold may be assumed to be in the branch locus). The Euler characteristic is $\chi(X_2) = 2\chi(\mathbb{CP}^3) - \chi(D) = -16$, hence $b_3(X_2) = 20$. The divisor D lifts isomorphically to a divisor D' on X_2 and calculation in cohomology (cf. [18], §4.4.) shows $-K_{X_2} = p^*[-D] - D' = 2[D'] - [D'] = [D']$, thus D' is an anticanonical divisor on X_2 (and so every smooth quartic occurs in the anticanonical class of some X_2). Further, D' has degree two in X_2 , so $-K_{X_2}^3 = 8 \cdot 2$, the extra factor 2 coming from the degree of the covering. Then the topology of a G_2 -manifold M constructed from X_2 and any Fano of the previous example (or X_2 again) is obtained by Theorem 8.57, e.g. if M is constructed from X_2 and $\mathbb{CP}^2 \times \mathbb{CP}^1$ then $b_2(M) = 0$, $b_3(M) = 117$.

Example 8.60 (Fano 3-folds as divisors). A quadric hypersurface $Q \subset \mathbb{CP}^4$ is a Fano 3-fold with $b_2(Q) = 1$, $b_3(Q) = 0$. The value of b_2 is implied by Lefschetz hyperplane theorem and then b_3 is again determined by the Euler characteristic. For the latter, recall that we may express the full Chern class of a projective hypersurface (and more generally,

of a complete intersection of hypersurfaces) knowing that the degree of its normal line bundle equals the degree of hypersurface by the adjunction formula. Then, writing x for the pull-back of the generator of the cohomology ring of the ambient \mathbb{CP}^4 we find $c(Q) = (1+x)^5(1+2x)^{-1} = 1+3x+4x^2+2x^3$, hence $\chi(Q) = \langle 2x^3, [Q] \rangle = 4$. The computation $c_1(Q) = 3x$ also tells us that a K3 surface D in $|-K_Q|$ is obtained as complete intersection of Q with a cubic hypersurface $X_3 \subset \mathbb{CP}^4$, so $\deg(D) = 6$. Conversely, for any given sextic K3 surface D in \mathbb{CP}^4 , consider the linear systems of quadric and cubic 3-folds containing D . We can find *non-singular* quadric and cubic 3-folds in these linear systems by Bertini's theorem [18]. Note that this proves Theorem 6.45 for the deformation class of Q , including the surjectivity of the projection π . Now $-K_Q^3 = -\chi(C)$ and notice that a curve $C = D \cdot D$ is obtained by intersecting D with a generic cubic hypersurface, so $\deg(C) = 18$ and $c(C) = (1+x)^5(1+2x)^{-1}(1+3x)^{-2} = 1-3x$, giving $-K_Q^3 = 54$.

A cubic $X_3 \subset \mathbb{CP}^4$ is also a Fano 3-fold and similar arguments show that every sextic K3 surface D occurs in $|-K_{X_3}|$ for some smooth X_3 , by intersecting with a suitable quadric hypersurface. We calculate, by the above method, that $b_2(X_3) = 1$, $b_3(X_3) = 10$, and $-K_{X_3}^3 = 24$.

The calculations are simplified for Fano 3-folds V whose anticanonical sheaf is very ample, which means that the anticanonical linear system of V defines an *embedding* making V into a smooth algebraic variety in \mathbb{CP}^{g+1} , where g is the genus of V . Then the anticanonical bundle of V is the restriction of $\mathcal{O}(1)$ and any K3 surface $D \in |-K_V|$ is a (generic) hyperplane section of V . Further, in this case,

$$-K_V^3 = 2g - 2 = \deg V = \deg D,$$

so $D \subset \mathbb{CP}^g$ is an algebraic K3 surface of degree $2g - 2$. Of course, a curve $C = D \cdot D$ is just the intersection of V with two generic hyperplanes.

A Fano 3-fold is called prime if the class $c_1(V)$ is primitive and generates the cohomology lattice $H^2(V)$ (then necessarily $b^2(V) = 1$). By Theorem 8.57 the two prime Fanos determine M uniquely up to a diffeomorphism.

Example 8.61 (Complete intersections). A smooth complete intersection $V = X_8$ of three quadric hypersurfaces in \mathbb{CP}^6 is a prime Fano 3-fold of degree 8 and genus 5. The prime property can be seen by first calculating the full Chern class (cf. Example 8.60) $c(X_8) = (1+x)^7(1+2x)^{-3} = 1+x+3x^2-3x^3$. It follows that $\chi(X_8) = -24$ and as $b_2(X_8) = 1$ by Lefschetz, then $b_3(X_8) = 28$. A hyperplane section $D \in |-K_{X_8}|$ is generically a smooth octic K3 surface in \mathbb{CP}^5 . Applying Bertini's theorem, we see that every octic K3 surface occurs in the anticanonical divisor on some (smooth) X_8 .

Another similar example of prime Fano 3-fold $V = X_6$ is given by a smooth complete intersection of a quadric and a cubic in \mathbb{CP}^5 . It has genus 4 and degree 6, with $b_2(V) = 1$, $b_3(V) = 40$.

Example 8.62 (Prime Fano 3-fold of genus 12). This prime Fano 3-fold X_{22} , a smooth variety of degree 22 in \mathbb{CP}^{13} , was originally found by Iskovskikh [21] and further investigated by Mukai. See [31] and references therein for various descriptions of X_{22} . A 3-fold

X_{22} is not isomorphic to a branched double cover of any homogeneous space or a complete intersection of hypersurfaces in a homogeneous space. It is known that X_{22} is a compactification of \mathbb{C}^3 and has the homology of $\mathbb{C}P^3$, but admits a 6-dimensional family of deformations. By the remarks before Example 8.61, an anticanonical divisor on any X_{22} defines a point in $\mathcal{H}\mathcal{S}(22)$. In the case of $V = X_{22}$ Theorem 6.45 reproves Theorem 7 of [12]. As $-K_{X_{22}}^3 = 22$ Theorem 8.57 shows that using X_{22} instead of $\mathbb{C}P^3$ one obtains G_2 -manifolds with ‘smaller’ homology and fundamental group.

The G_2 -manifolds constructed from any two of the Fano 3-folds described above satisfy $71 \leq b_3(M) \leq 155$, by Theorem 8.57. The upper bound is attained by constructing M from a pair of $\mathbb{C}P^3$ ’s and the lower bound from a pair of prime X_{22} ’s.

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